1/7/21 Lecture outline

⋆ Reading: Schwarz chapter 14 (Path integrals).

• Now introduce **sources** $J(x)$ for the field $\phi(x)$ as a trick to get the time order products from derivatives of a generating functional. Let's record, for future use, the needed gaussian integrals:

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_i + \tilde{J}_i\phi_i) = \pi^{N/2}(\det B)^{-1/2} \exp(B^{-1}_{ij}\tilde{J}_i\tilde{J}_j/4).$$

Evaluate via completing the square: the exponent is $-(\phi, B\phi) + (\tilde{J}, \phi) = -(\phi', B\phi') + \frac{1}{4}(\tilde{J}, B^{-1}\tilde{J})$, where $\phi' = \phi - \frac{1}{2}B^{-1}\tilde{J}$. Again, we can similarly evaluate Gaussian integrals with phases in the exponent by analytic continuation

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(\frac{i}{\hbar}(\tfrac{1}{2}A_{ij}\phi_i\phi_i + J_i\phi_i)) = (2\pi i\hbar)^{N/2}(\det A)^{-1/2} \exp(-iA^{-1}_{ij}J_iJ_j/2\hbar).$$

replacing $B \to -i(\frac{1}{2}A + i\epsilon)/\hbar$ and redefining $\tilde{J} \to iJ/\hbar$ for later convenience.

• Consider e.g. QM with Hamiltonian $H(q, p)$, modified by introducing a source for q, $H \to H - J(t)q$. (We could also add a source for p, but don't bother doing so here.) Consider moreover replacing $H \to H(1 - i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_J = \int [dq] \exp[i \int dt(L + J(t)q)/\hbar] \equiv Z[J(t)].$$

Once we compute $Z[J(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[J]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0| \prod_{j=1}^{n} Tq(t_j)|0\rangle = \prod_{j=1}^{n} \frac{1}{i}\frac{\delta}{\delta J(t_j)} Z[J]\big|_{J=0},$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture. We'll be interested in such generating functionals, and their generalization to quantum field theory (replacing $t \to (t, \vec{x})$).

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta J(t)}J(t') = \delta(t - t')$.

• We'll want to compute amplitudes like

$$\frac{\langle 0|\prod_i Tq(t_i)|0\rangle_{J=0}}{\langle 0|0\rangle_{J=0}}$$

and for these the $\det A$ factor in the Gaussian integrals will cancel between the numerator and the denominator. This is related to the cancellation of vacuum bubble diagrams.

• Let's apply the above to compute the generating functional for the example of QM harmonic oscillator (scaling $m=1$),

$$Z[J(t)] = \int [dq(t)] \exp(-\frac{i}{\hbar}\int dt \left[\tfrac{1}{2}q(t)(\frac{d^2}{dt^2}+\omega^2)q(t) - J(t)q(t)\right]).$$

This is analogous to the multi-dimenensional gaussian above, where i is replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix A_{ij} is replaced with the differential operator $A \to -(\frac{d^2}{dt^2}+\omega^2 - i\epsilon)$, where the $i\epsilon$ is to damp the gaussian, as mentioned above. Doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want, so

$$\frac{\langle 0|0\rangle_J}{\langle 0|0\rangle_{J=0}} = \text{“}\exp[-i\tfrac{1}{2}A^{-1}_{ij}J_iJ_j/\hbar]\text{”} = \exp[-\tfrac{1}{2}\hbar\int dtdt' J(t)G(t-t')J(t')],$$

with $G(t)$ the Green's function for the oscillator, $(-\partial_t^2 - \omega^2 + i\epsilon)G(t) = i\delta(t)$,

$$G(t) = \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\frac{i \quad e^{-iEt/\hbar}}{E^2/\hbar^2 - \omega^2 + i\epsilon} = \frac{1}{2\omega}e^{-i\omega|t|}. \tag{1}$$

The $i\epsilon$ here does the same thing as in the Feynman propagator: the pole at $E=\hbar\omega$ is shifted below the axis and that at $E=-\hbar\omega$ is shifted above. Equivalently, we can replace $E\to E(1+i\epsilon)$, to tilt the integration contour below the $-\omega$ pole and above the $+\omega$ pole. Note then that $e^{-iEt/\hbar}\to e^{-iEt/\hbar}e^{Et\epsilon/\hbar}$, which projects on to the vacuum for $t\to\infty$ (the $i\epsilon$ projects on to the vacuum in the far future and also the far past).

For $t>0$, the E contour is closed in the LHP and the residue is at $E=\hbar\omega$, while for $t<0$ the contour is closed in the UHP, with residue at $E=-\hbar\omega$.

• Now that we know the generating functional, we can use it to compute time ordered expectation values via

$$\langle 0|T\prod_{i=1}^{n}\phi_H(t_i)|0\rangle/\langle 0|0\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^{n}\phi(t_i)\exp(iS/\hbar) = Z_0^{-1}\prod_{i=1}^{n}\frac{\hbar}{i}\frac{\delta}{\delta J(t)}|_{J=0}.$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). E.g.

$$\langle \phi_b(\vec{x}, T) | e^{-iHT} | \phi_a(\vec{x}, 0) \rangle = \int [d\phi] e^{iS/\hbar} \qquad S = \int d^4x \mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle \Omega | T \prod_{i=1}^{n} \phi_H(x_i) | \Omega \rangle = Z_0^{-1} \int [d\phi] \prod_{i=1}^{n} \phi(x_i) \exp(iS/\hbar),$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$. Again, as noted above, the T ordering will be automatic.

• On to QFT and the Klein-Gordon theory,

$$Z_0 = \int [d\phi] e^{iS/\hbar} \qquad S = \tfrac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term. This is completely analogous to our QM SHO example, simply replacing $\frac{d^2}{dt^2} + \omega^2 - i\epsilon$ there with $\partial^2 + m^2 - i\epsilon$ here – again, the $i\epsilon$ is to make the oscillating gaussian integral slightly damped. I.e. we should take $S = \frac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $A \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar: it's the Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

As in the SHO QM example, we can compute field theory Green's functions via the generating functional

$$Z[J(x)] = \int [d\phi] \exp(i \int d^4x [\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0 | T \prod_{i=1}^{n} \phi(x_i) | 0 \rangle / \langle 0 | 0 \rangle = Z[J]^{-1} \prod_{j=1}^{n} \left(-i \frac{\delta}{\delta J(x_i)} \right) Z[J] \big|_{J=0}.$$

E.g. for the KG example, $A = (-\partial^2 - m^2 + i\epsilon)$, so the generating functional is

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1} \int d^4x d^4y J(x) D_F(x-y) J(y)), \tag{2}$$

with $(-\partial^2 - m^2 + i\epsilon) D_F(x-y) = i\delta^4(x-y)$,

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}.$$

Can use this generating function to compute free field time ordered products, it reproduces Wick's theorem, Feynman diagrams. I'll discuss this soon, including interactions to make it more interesting. For now note that $\langle T\phi(x)\phi(y)\rangle = G^{(2)}(x-y) = D_F(x-y)$.

• It is often useful to consider QFT in Euclidean spacetime, and then rotate the answers back to Minkowski space. Note that $L_E = -H$ and $e^{iS/\hbar} \to e^{-\int dt_E H/\hbar}$, so the Euclidean path integral gives a partition function – it is literally the partition function if Euclidean time is compactified $t_E \sim t_E + \hbar\beta$ with $\beta = 1/k_B T$. Let's discuss this in a bit more detail, and we'll return to it later – e.g. when we compute loop integrals. We want to analytically continue all energies by $+\pi/2$ – this is called a Wick rotation. Thanks to the pole placement in the propagators (the $i\epsilon$), this continuation does not cross any poles, and now we integrate k_0 up the imaginary axis. Now we define $k_0 = ik_4$ with k_4 real, and then $k^2 = -k_E^2$ and $d^4k = id^4k_E$. Our Fourier transforms have e^{ikx} and we don't want that to blow up, so we analytically continue all x_0 by $-\pi/2$. Then we change variables via $x_0 = -ix_4$ with x_4 real, so $d^4x = -id^4x_E$. We then see that the Wick rotation takes $S = \int d^4x \mathcal{L} \to (-i)\int d^4x_E(-\mathcal{H}) = i\int d^4x_E \mathcal{H}$, so $e^{iS/\hbar} \to e^{-\int d^4x_E \mathcal{H}}$ which is copacetic since the exponential is damped.

• Let's now consider the path integral for a free, massive, Dirac Fermion. The functional integral is over Grassmann valued fields. Consider a Grassmann coordinate θ. It anticommutes with any other Grassmann coordinate, including itself, so it sqaures to zero. Taylor expansions are thus simple, e.g. $e^{a\theta} = 1 + a\theta$. The constraints on what we want integration to satisfy (e.g. linearity) ends up requiring that $\int d\theta 1 = 0$ and $\int d\theta\theta = 1$, i.e. integration behaves the same as differentiation. We'll be interested in complex θ and θ^* and then $\int d\theta^* d\theta \exp(-\theta^* b\theta) = b$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j) = \det B$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j)\theta_k\theta_l^* = (B^{-1})_{kl} \det B$.

• We can introduce sources for the fields:

$$Z[\bar\eta_i,\eta_i] = \int d\bar\theta_i d\theta_i \exp(i(A_{ij}\bar\theta_i\theta_j + \bar\eta_i\theta_i + \bar\theta_i\eta_i])$$
$$= \int d\bar\theta_i d\theta_i (1 + i(\bar\theta, A\theta))(1 + i\bar\eta\theta)(1 + i\bar\theta\eta),$$
$$= -i \det A \exp(-i\bar\eta_i A^{-1}_{ij}\eta_j).$$

• Generalize to functional integrals over fermionic fields;

$$Z[\bar\eta,\eta] = \int [d\bar\psi][d\psi] \exp(i \int d^4x[\bar\psi(i\partial\!\!\!/ - m)\psi + \bar\eta\psi + \bar\psi\eta]$$
$$= Z_0 \exp[-\int d^4x d^4y \bar\eta(x) S_F(x-y)\eta(y).$$

where

$$S_F[x-y] = i(i\partial\!\!\!/ - m)^{-1} = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k\!\!\!/ - m + i\epsilon}.$$

Get e.g.

$$\langle 0|T\psi(x)\bar\psi(y)|0\rangle = Z_0^{-1}(-i\frac{\delta}{\delta\bar\eta(x)})(i\frac{\delta}{\delta\eta(y)})Z[\eta,\bar\eta]|_{\eta,\bar\eta=0} = S_F(x-y).$$

We will see how this gives the Feynman rules for fermions that you saw last quarter. The $\det B$ in the numerator instead of the denominator will be related to the fact that every closed fermion loop gets an extra -1 factor. (This relative minus sign is put to good use with supersymmetry!)

• Comments about functional integrals for gauge fields. Emphasize that gauge invariance is *not* really a symmetry: it is a redundancy in our description. Configurations differing by gauge transformations are to be interpreted as *the same* physical state. This shows up in the functional integral in that we should not even be integrating over gauge equivalent configurations.

Recall gauge invariance, $A = A_\mu dx^\mu \to A^\alpha = A + d\alpha(x)$, with $\psi_i \to e^{-iq_i\alpha(x)}\psi$. Redundancy in description, can only observe gauge invariant quantities. Need to replace $\partial_\mu\psi_i \to D_\mu\psi_i \equiv (\partial_\mu + iq_iA_\mu)\psi_i$. Then $D^\alpha_\mu\psi^\alpha_i = e^{-iq_i\alpha}D_\mu\psi_i$ transforms nicely, with just an overall phase, and $\bar\psi_i D_\mu\psi_i$ is gauge invariant. So the Dirac lagrangian, $\bar\psi(iD\!\!\!\!/ - m)\psi$ is gauge invariant. The terms linear in A_μ give $\mathcal{L} \supset -A_\mu j^\mu$, with j^μ the conserved current.

In the functional integral, will have $\int[dA]\exp(iS)$. Integration measure must be gauge invariant, implies it gets a factor of gauge orbit volume. Would like to integrate only over a slice of inequivalent gauge fields, without integrating over the gauge orbits. Need to do this, since otherwise there is no well defined B^{-1}. Recall $S = \int d^4x[-\frac{1}{4}F_{\mu\nu}^2] = \frac{1}{2}\int d^4k A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x)$. Note action vanishes if $\tilde{A}_\mu(k) = k_\mu\alpha(k)$. Gauge invariance. $A_\mu^T = P_{\mu\nu}A^\nu$, $P_{\mu\nu} = g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2$. $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A_\mu^T\partial^2 g^{\mu\nu}A_\nu^T$. Can't invert kinetic terms uniquely to find Green's function. It's useful to fix the gauge, and then check at the end that the gauge fixing didn't matter.

The functional integral should be over $\int[dA^\mu]/(GE)$, where we divide by the volume of the gauge equivalent orbits. The gauge equivalent orbits are associated with gauge transformations $\alpha(x)$, e.g. $A_\mu \to A_\mu + \partial_\mu\alpha(x)$ in the Abelian case. We want to do the functional integral over A_μ, dividing out by the $\partial_\mu\alpha(x)$ shifts.

• Seeing contact terms from the functional integral. Recall that the classical EOM are satisfied in correlation functions modulo contact terms, e.g. $(-\partial_x^2 - m^2 + i\epsilon)\langle T\phi(x)\phi(y)\rangle = i\delta^4(x-y)$. This can be nicely seen from the path integral:

$$\langle\phi(x)\rangle_J = \frac{1}{Z[0]}\int[d\phi]e^{iS/\hbar + i\int J\phi}\phi(x) = \frac{1}{Z[0]}\int[d\phi]e^{\frac{i}{\hbar}S[\phi+\delta\phi]+i\int J(\phi+\delta\phi)}(\phi(x)+\delta\phi(x)),$$

where in the last equation we redefined the integration variable from $\phi \to \phi + \delta\phi$. If we expand in $\delta\phi$, the $\delta\phi^0$ term already agrees with the LHS, so the $\delta\phi^1$ term must give zero. You can check that, upon e.g. taking $-i\frac{\delta}{\delta J(y)}$, this gives the classical EOM with an extra contact term. Such equations are called the Schwinger-Dyson differential equations.

with $\phi(x) \to \frac{1}{i}\frac{\delta}{\delta J(x)}$. Either way gives the same answers for the green's functions, of course, – it's just semantics for what we want to call the source.

• Now let's consider an interacting theory. Notice that

$$\int[d\phi]\exp(\frac{i}{\hbar}[S_{free} + S_{int}[\phi] + \hbar\int d^4xJ\phi]) = \exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}])Z_{free}[J].$$

So

$$Z[J] = N\exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}])Z_{free}[J], \tag{3}$$

where N is an irrelevant normalization factor (independent of J). The green's functions are then given by

$$\begin{aligned} G^{(n)}(x_1\ldots x_n) &= \frac{\int[d\phi]\phi(x_1)\ldots\phi(x_n)\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}{\int[d\phi]\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}, \\ &= \frac{1}{Z[J]}\prod_{j=1}^n\left(-i\hbar\frac{\delta}{\delta J(x_j)}\right)\cdot Z[J]\big|_{J=0}. \end{aligned}$$

(The denominator (in both lines) cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function.)

• Illustrate the above formulae, and relation to Feynman diagrams, e.g. $G^{(1)}$, $G^{(2)}$ and $G^{(4)}$ in $\lambda\phi^4$ theory. The functional integral accounts for all the Feynman diagrammer; even symmetry factors etc. come out simply from the derivatives w.r.t. the sources, and the expanding the exponentials,

$$G^{(n)}(x_1,\dots x_n) = \frac{1}{Z[J]}\prod_{j=1}^{n}(-i\frac{\delta}{\delta J(x_j)})\sum_{N=1}^{\infty}\frac{1}{N!}\left(-i\frac{\lambda}{4!\hbar}\int d^4y(-i)^4\frac{\delta^4}{\delta J(y)^4}\right)^N Z_0[J]\big|_{J=0}.$$

etc. Consider, for example, the 4-point function $G^{(4)}(x_1,x_2,x_3,x_4)\equiv\langle T\phi(x_1)\dots\phi(x_4)\rangle/\langle 0|0\rangle$ in $\frac{\lambda_4}{4!}\phi^4$. So take 4-fuctional derivatives w.r.t. the source, at points $x_1\dots x_4$, i.e. $\delta/\delta J(x_1)\dots\delta/\delta J(x_4)$. The $\mathcal{O}(\lambda^0)$ term thus comes from expanding the exponent in (2) to quadratic order (4 J's), corresponding to the disconnected diagrams with two propagators. Each propagator ends on a point x_i. This is like the 4-point function in the SHO homework. Now consider the $\mathcal{O}(\lambda)$ contribution, coming from expanding out the interaction part of the exponent in (3) to $\mathcal{O}(\lambda)$. There are now 4 extra $\delta/\delta J(y)$, for a total of 8, so the contributing term comes from expanding the exponent in (2) to 4-th order, i.e. there are 4 propagators. This gives the connected term, along with several disconnected terms.