1/5/21 Lecture outline

⋆ Reading: Schwarz chapter 14 (Path integrals).

• Classical mechanics is usually taught by first introducing the Lagrangian description, and then the Hamiltonian description. The Lagrangian description is usefully phrased in terms of the principle of least action, but action itself is not a classical observable and there is something mysterious about this path-dependent functional, which happens to have the same units as $\hbar$. A nice aspect of the principle of least action is that we do not have to pick a time slicing, and the action exhibits the full symmetries, e.g. Lorentz invariance and, in general relativity, the full general coordinate transformation invariance. Of course the Hamiltonian description is also useful, with its emphasis on phase space, Poisson brackets, canonical transformations, the Hamilton generating time translations via Hamilton's equations, seeing conserved charges, etc.

Quantum mechanics is usually taught first by quantizing the Hamiltonian description, replacing the phase space dynamical quantities with operators and replacing Poisson brackets with equal time commutators (or anticommutators for Fermions). An alternative description in terms of the principle of least action was developed by Feynman, with prescient early hints in Dirac's classic book on QM in the form of the equation $\psi \sim e^{iS/\hbar}$. Feynman came to this approach by intuitively thinking about double slit interference and realizing that empty space can be thought of as being filled with screens that are full of holes, so such interference and taking multiple paths is always there. The path integral generalizes immediately from QM to QFT, and for different types of fields. Unlike canonical quantization, it makes Lorentz and Poincare symmetry manifest, and also gives a way to define QFT beyond perturbation theory. The classical limit is clarified, as the stationary phase limit of an integral, much as with light rays in geometric optics; this also gives a nice perspective on the WKB approximation. The path integral also helps to connect QFT with statistical physics: the partition function $\mathrm{Tr}e^{-\beta H}$ can be considered as a path integral with time t replaced with Euclidean time τ and compactified on a circle $\tau \sim \tau + \beta$.

In canonical quantization, we compute expectation values of operators. In the path integral description, there are no operators and time-ordering comes out automatically.

The path integral is great conceptually, but not used much in QM classes because of the math. Least action only requires learning functional differentiation. Feynman's path integral is based on functional integration, and mathematicans had not yet developed that much (it was partly developed by Norbert Wiener $\sim$ 1921 in the context of Brownian motion; Wiener was a professor at MIT, and maybe Feynman heard about it there when he was an undergraduate). As we will see, for free fields the action is quadratic in the fields, and the functional integrals are thus Gaussian and easy to evaluate, and then we can add perturbations and re-obtain the Feynman rules for computing amplitudes.

• Recall canonical quantization for a scalar (for simplicity) field $\phi(x)$ in D spacetime dimensions. The conjugate momentum is $\Pi = \partial\mathcal{L}/\partial\dot{\phi}$ and we promote them to operators satisfying

$$[\phi(t,\vec{x}), \Pi(t,\vec{x}')] = i\hbar\delta^{D-1}(\vec{x}-\vec{x}').$$

Quantum mechanics is recovered for $D=1$, and we'll set $\hbar = 1$. The S-matrix elements, used to compute scattering cross sections and lifetimes etc. are computed from an amplitude $\langle f|S|i\rangle$ which is related to the vacuum expectation values of time-ordered products of the fields. This is seen from Dyson's formula or from the LSZ derivation:

$$\begin{aligned}\langle f|i\rangle &= \langle k_{1'}\dots k_{n'}|k_1\dots k_n\rangle \\ &= i^{n+n'}\prod_{j'=1}^{n'}\int d^4x'_j e^{ik'_jx'_j}(\partial_{j'}^2+m^2)\prod_{j=1}^{n}e^{-ik_jx_j}(\partial_j^2+m^2)G_{n+n'}(x_1\dots x_n, x_{1'}\dots x_{n'}),\end{aligned} \tag{1}$$

where G is called the Green's function and defined by

$$G_{n+n'}(x_1\dots x_n, x_{1'}\dots x_{n'}) \equiv \langle 0|T\phi(x_{1'})\dots\phi(x_{n'})\phi(x_1)\dots\phi(x_n)|0\rangle. \tag{2}$$

Using Wick's theorem,

$$T(\phi_1\dots\phi_n) =: \phi_1\dots\phi_n : + : \text{all contractions} :,$$

gives Feynman's rules for computing amplitudes, from Feynman diagrams.

• Let's first introduce the path integral in QM (QFT in $d = 1+0$ spacetime dimensions). The probability amplitude to go from ϕ at time t to ϕ' at time t' is

$$_H\langle\phi', t'|\phi, t\rangle_H = U(\phi,\phi';T) = \langle\phi'|e^{-iHT/\hbar}|\phi\rangle.$$

Satisfies SE $i\hbar\partial_T U = HU$. Feynman:

$$U(\phi,\phi';T) = \int [d\phi(t)]e^{iS[\phi(t)]/\hbar},$$

where the integral is over all possible paths with the prescribed boundary conditions at t and t'. In QFT we will be interested in S-matrices where we can take the initial and final times to $\mp\infty$ so we don't need to worry about imposing boundary conditions on the paths.

The functional integral can be broken into time slices, as way to define it. E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2}\int\prod_{i=1}^{N-1}d\phi_i\exp[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(\phi_i-\phi_{i-1})^2]$$

Where we take $\epsilon\to 0$ and $N\to\infty$, with $N\epsilon=T$ held fixed. Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty}d\phi\exp(ia\phi^2/2)=\sqrt{\frac{2\pi i}{a}},\qquad \int_{-\infty}^{\infty}d\phi\exp(i(\tfrac{1}{2}a\phi^2-J\phi))=\sqrt{\frac{2\pi i}{a}}\exp(-iJ^2/2a).$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a)>0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a\to a+i\epsilon$, with $\epsilon>0$, and then take $\epsilon\to 0^+$. We'll see that this is related to the $i\epsilon$ in the Feynman propagator, which gave the T ordering.

After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2}\exp[\frac{m}{2\pi i\hbar n\epsilon}(\phi_n-\phi_0)^2].$$

So the final answer is

$$U(x_b,x_a;T)=\left(\frac{2\pi i\hbar T}{m}\right)^{-1/2}\exp[im(\phi'-\phi)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.) Consider the phase of Uas a function of $\Delta\phi=\phi'-\phi$, fixed T. For large $\Delta\phi$, nearly constant wavelength λ, and we can recover $p=\hbar k$, with $p=\partial S_{cl}/\partial\phi'$, and likewise can get $E=\hbar\omega$ from $E=-\partial S_{cl}/\partial t'$.

• Nice application: Aharonov-Bohm. Recall $L=\frac{1}{2}m\dot{\vec{x}}^2+q\dot{\vec{x}}\cdot\vec{A}-q\phi$. Solenoid with $B\neq 0$ inside, and $B=0$ outside. Phase difference in wavefunctions is

$$e^{i\Delta S/\hbar}=e^{iq\oint\vec{A}\cdot d\vec{x}/\hbar}=e^{iq\Phi/\hbar}.$$

Aside on Dirac quantization for magnetic monopoles.

• Can derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t'|q, t\rangle = \int\int \prod_{j=1}^{N} dq_j \langle q'|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \dots \langle q_1|e^{-iH\delta t}|q\rangle,$$

where we'll take $N \to \infty$ and $\delta t \to 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\dots}$, we're not going to have to worry about this for $\delta t \to 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-i\delta t p^2/2m}|p_1\rangle\langle p_1|e^{-iV(q)\delta t}|q_1\rangle,$$

$$= \int dp_1 e^{-iH(p_1,q_1)\delta t} e^{ip_1(q_2-q_1)}.$$

This leads to the Legendre transformation from the Hamiltonian to the Lagrangian in the exponent since

$$\langle q', t'|q, t\rangle = \int [dq(t)][dp(t)] \exp(i\int_t^{t'} dt(p(t)\dot{q}(t) - H(p,q))),$$

and taking H quadratic in momentum and doing the p gaussian integral recovers the Feynman path integral.

• The same derivation as above leads to e.g.

$$\langle q_4, t_4|T\widehat{q}(t_3)\widehat{q}(t_2)|q_1, t_1\rangle = \int [dq(t)] q(t_3)q(t_2)e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.