215b Homework 1, due 1/12/21

1. In class, I sketched how to compute $U(x_b, t_b; x_a, t_a) = \langle x_b | e^{-iH(t_b - t_a)/\hbar} | x_a \rangle$ using the path integral, $U = \int [dx] \exp(iS/\hbar)$ for a free-particle. The result was given in class.

(a) The result for this case of a free-particle happens to be of the form $U = F(t_b - t_a) e^{iS_{cl}/\hbar}$. Verify this, by computing $S_{cl}$ for the free particle. Verify also that $p = \partial S_{cl}/\partial x_b$ and $E = -\partial S_{cl}/\partial t_b$.

(b) Re-derive the expression given in class for $U = \langle x_b | e^{-i\widehat{H}(t_b - t_a)/\hbar} | x_a \rangle$, for the case of a free-particle, by using the standard operator description of QM with $\widehat{H} = \widehat{p}^2/2m$.

2. (taken from Peskin and Schroeder, problem 9.2)

(a) Evaluate the quantum statistical partition function

$$Z = \mathrm{Tr} e^{-\beta H}$$

using the strategy which led to the path integral (introducing lots of complete sets of states) for evaluating the matrix elements of $e^{-iHt/\hbar}$ in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length $\beta$ and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

(b) Evaluate this integral for the simple harmonic oscillator, $L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on $\beta$ is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on $\beta$ in any discretization. However, the dependence on $\omega$ should be unambiguous. Show that up to a (possibly divergent and $\beta$ dependent) constant the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n\pi)^2}\right)$$

useful.]

(c) In class we discussed how

$$\langle 0|0\rangle_f = \int [dq] \exp[i \int dt (L + f(t)q)/\hbar] \equiv Z[f(t)].$$

$Z[f]$ is a generating functional for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0| \prod_{j=1}^{n} Tq(t_j)|0\rangle = \prod_{j=1}^{n} \frac{1}{i} \frac{\delta}{\delta f(t_j)} Z[f]\big|_{f=0}, \tag{1}$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture. We consider the harmonic oscillator in quantum mechanics ("SHO"), and motivated the result

$$Z_{SHO}[f] = \langle 0|0\rangle_f = \exp[\frac{i}{2} \int_{-\infty}^{\infty} dtdt' f(t)G(t-t')f(t')], \tag{2}$$

with (setting $\hbar = 1$)

$$G_{SHO}(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2 + \omega^2 - i\epsilon} = \frac{i}{2\omega} e^{-i\omega|t|}. \tag{3}$$

Verify explicitly that, using (1) and (2) it follows that e.g.

$$\langle 0|Tq(t_1)q(t_2)|0\rangle = -iG_{SHO}(t_2 - t_1), \tag{4}$$

and

$$\langle 0|Tq(t_1)q(t_2)q(t_3)q(t_4)|0\rangle = (-i)^2 \left(G_{SHO}(t_2 - t_1)G_{SHO}(t_3 - t_4) + \text{perms}\right) \tag{5}$$

where perms means two similar terms, with $t_2 \leftrightarrow t_3$ and $t_2 \leftrightarrow t_4$.

3. Schwartz 14.1: show that for complex scalar fields

$$\int [D\phi] \exp(i \int d^4x(\phi^* M\phi + J\phi + J^*\phi^*)) = \mathcal{N}(\det M)^{-1} \exp(iCJ^* M^{-1} J)$$

where my $[D\phi]$ notation means the same as his integration measure, and $\mathcal{N}$ is some infinite constant that we don't bother to compute because it'll cancel in physical quantities, and $C$ is some constant that you should determine.

4. Repeat the previous exercise with $\phi \to \psi$, where $\psi$ is a Fermion field, determining how the RHS is modified (using the rules of Grassmann integration).