1/27/20 Lecture outline

⋆ **Reading: Sections 10.1, 10.2, 10.3, 10.4**

• Last time: the center of mass and rotation. Consider a collection of masses m_a, or a mass distribution $\rho(\vec{r})$. The total mass is $M = \sum_a m_a = \int dV \rho(\vec{r})$, where we use either a sum or an integral as appropriate, and we can convert between them via e.g. $\rho(\vec{r}) = \sum_a m_a \delta^3(\vec{r} - \vec{r}_a(t))$. The total momentum is $\vec{P} = \sum_a m_a \dot{\vec{r}}_a(t) = \int dV \rho(\vec{r}) \frac{d\vec{r}}{dt}$. Write $\vec{P} = M\dot{\vec{R}}$ where $\vec{R}$ is the center of mass (or center of momentum) position $\vec{R} \equiv \frac{1}{M}\sum_a m_a \vec{r}_a$ or $\vec{R} = \frac{1}{M}\int \vec{r} dm$, where $dm \equiv \rho(\vec{r})dV$. Using Newton's law, $\vec{F}_{ext} = \dot{\vec{P}} = M\frac{d^2\vec{R}}{dt^2}$. If $\vec{F}_{ext} = 0$, then the CM will move at constant velocity; we saw this in the two-body central force section where $\vec{F}_{ext} = 0$ and we took $\vec{R} = 0$. Now define $\vec{r}'_a$ by $\vec{r}_a = \vec{R} + \vec{r}'_a$. Here $\vec{r}_a$ is taken to be a vector in an inertial reference frame with a fixed origin, and $\vec{r}'_a$ is the position relative to an origin at the CM. Note that $\sum_a m_a \vec{r}'_a = 0$.

• The angular momentum relative to the fixed origin is $\vec{L} = \sum_a \vec{r}_a \times m_a \dot{\vec{r}}_a = \vec{R} \times \vec{P} + \sum_a \vec{r}'_a \times m_a \dot{\vec{r}}_a{}'$, where two terms drop out thanks to $\sum_a m_a \vec{r}'_a = 0$ and its derivative. This shows that the total angular momentum is that of the CM plus that relative to the CM. Now $\frac{d}{dt}\vec{R} \times \vec{P} = \vec{R} \times \dot{\vec{P}} = \vec{R} \times \vec{F}^{ext} = \vec{\Gamma}^{ext}$, the external torque acting on the CM. Likewise $\frac{d}{dt}\sum_a \vec{r}'_a \times \vec{p}_a = \sum_a \vec{r}'_a \times \vec{F}^{ext}_a = \vec{\Gamma}^{ext}|_{CM}$, the external torque relative to the CM.

The total kinetic energy is $T = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a{}^2 = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a{}'^2$.

• For rotation around a fixed axis, we replace $\dot{\vec{r}}_a = \vec{\omega} \times \vec{r}_a$. Then $T_{rot} = \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a{}'^2 = \frac{1}{2}\sum_a m_a(\omega^2 r'_a{}^2 - (\vec{\omega}\cdot\vec{r}'_a)^2) = \frac{1}{2} I_{jk}\omega_j\omega_k$ where $I_{jk} \equiv \sum_a m_a(\vec{r}_a^2 \delta_{jk} - r_j r_k) = I_{kj}$ (so $I = I^T$).

• The moment of inertia tensor I_{jk} also enters in $L^{rot}_j = \sum_k I_{jk}\omega_k$, where $\vec{L}^{rot}$ is the CM rotational angular momentum: $\vec{L}^{rot} = \sum_a \vec{r}'_a \times m_a(\vec{\omega} \times \vec{r}'_a) = \sum_a m_a(\vec{\omega} r'_a{}^2 - \vec{r}'_a(\vec{\omega}\cdot\vec{r}'_a))$.

E.g. take $\vec{\omega} = \omega\hat{z}$, then $\vec{v}_a = \omega \times \vec{r}_a = -\omega y_a \hat{x} + \omega x_a \hat{y}$ and $\ell_a = m_a \vec{r}_a \times \vec{v}_a = m_a\omega(-z_a x_a \hat{x} - z_a y_a \hat{y} + (x_a^2 + y_a^2)\hat{z})$. The CM angular momentum thus has $\vec{L}_z = I_{zz}\omega$ where $I_{zz} \equiv \sum_a m_a \rho_a^2$, where $\rho_a^2 = x_a^2 + y_a^2$ is the distance of the point to the axis of rotation. The products of inertia enter in e.g. $L_x = I_{xz}\omega$ and $L_y = I_{yz}\omega$, with $I_{xz} = -\sum_a m_a x_a z_a$ and $I_{yz} = -\sum_a m_a y_a z_a$.

• Example: consider a wheel of radius R that is rolling without slipping with velocity $\vec{V} = V\hat{x}$. The center of the wheel has $y = R$, and we then find $\omega = -\omega\hat{z}$ with $\omega = V/R$. The velocity of a point on the wheel is $\vec{v} = \omega R\hat{x} + R\vec{\omega} \times \hat{r}$, where $\hat{r}$ points from the center of the wheel. For example, for the point of contact $\hat{r} = -\hat{y}$ and $\vec{v} = 0$, and for the top of the wheel $\vec{v} = 2\omega R\hat{x}$. A solid wheel has $L_z = I_{zz}\omega$, with $I_{zz} = \int dm \rho^2 = \frac{1}{2}MR^2$.

• Example: for a cube of side length a rotating around its center (so $\int dm \to \frac{M}{a^3}\int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz$), get $I_{jk} = \frac{1}{6}Ma^2\delta_{jk}$. If the cube is instead rotating around its corner (so the integrals are all $\int_0^a$ instead of $\int_{-a/2}^{a/2}$), can compute to get $I_{jk} = \frac{1}{6}Ma^2\delta_{jk} + \frac{1}{4}Ma^2(3\delta_{ij} - 1)$.

ended here

• Parallel axis theorem: replace $\vec{r}_a \to \vec{r}_a' = \vec{r}_a - \vec{d}$ (with $\sum_a m_a \vec{r}_a = 0$) for moment of inertia tensor for rotations about an axis displaced to $\vec{d}$. Get $I_{jk}(\vec{d}) = I_{jk}(0) + M(\vec{d}^2\delta_{jk} - d_j d_k)$. For example, for a solid wheel around a point on the rim this gives $I_{zz} = \frac{3}{2}MR^2$.

If the cube is instead rotating around a corner, take $\vec{d} = \frac{1}{2}a(1,1,1)$ and then get $I_{jk} = \frac{1}{6}Ma^2\delta_{jk} + \frac{1}{4}Ma^2(3\delta_{ij} - 1)$.

• The eigenvectors ω of the inertia tensor are called the principal axes, and the eigenvalues λ are called the principal moments: $\vec{L} = \lambda\vec{\omega}$. We can find three orthogonal eigenvectors $\vec{\omega}_{i=1,2,3}$ and write I in this basis as a diagonal matrix with the three eigenvalues $\lambda_{i=1,2,3}$ along the diagonal. For example, for a cube rotating around a corner, one of the principle axes is along the diagonal, so $\vec{\omega}_1 = \omega\frac{1}{\sqrt{3}}(1,1,1)$, which has principle moment eigenvalue $\lambda_1 = Ma^2/6$. The other two principle axes are perpendicular and here, because of the symmetry, they have the same eigenvalue, $\lambda_2 = \lambda_3$. The original I must have trace equal to $\lambda_1 + \lambda_2 + \lambda_3$ and determinant equal to $\lambda_1\lambda_2\lambda_3$ (since the diagonalized matrix of eigenvalues differs by a similarity transform $I \to R^{-1}IR$ and the trace and determinant are invariant under that. Indeed, find $\lambda_2 = \lambda_3 = \frac{11}{12}Ma^2$.

• We saw in the previous chapter that, for any vector $\dot{\vec{Q}}|_{space} = \dot{\vec{Q}}|_{body} + \vec{\omega} \times \vec{Q}$ where "space" refers to an inertial frame that is fixed in the lab, and "body" refers to a non-inertial frame that is fixed on the rotating body. Apply this to the case of angular momentum to get Euler's equation:

$$\frac{d\vec{L}}{dt}|_{space} = \vec{\Gamma}_{ext} = \frac{d\vec{L}}{dt}|_{body} + \vec{\omega} \times \vec{L}.$$

Use this and $L_j = I_{jk}\omega_k$ to determine the dynamical rotation $\vec{\omega}(t)$ of the body.