1/24/20 Lecture outline

$\star$ **Reading: Taylor Chapter 9, Sections 10.1, 10.2**

• Last time: free fall near earth's surface (we omit writing the prime on $\vec{r}'$)

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}, \qquad \vec{g} = \vec{g}_0 + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

where $\vec{g}$ is the observed free-fall acceleration, which includes the centrifugal force term. Note that $\Omega^2_{earth} R_{earth} \approx 3.38 \times 10^{-2} ms^{-2}$ is about a 0.3% correction to g. Take $\vec{r} \approx \vec{R}$ in $\vec{F}_{cf}$, which is a vector from the center of the earth to the position on the earth's surface where the experiment is done. Choose local coordinates near that location (assumed to be in the Northern hemisphere) such that $\hat{z}'$ points up (really it's $\hat{r}$) $\hat{y}'$ points North (it's really $-\hat{\theta}$), and $\hat{x}'$ points East (it's really $\hat{\phi}$). In this coordinate system $\vec{\Omega} = (0, \Omega \sin\theta, \Omega \cos\theta)$. This gives

$$\frac{d^2x}{dt^2} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta), \ \frac{d^2y}{dt^2} = -2\Omega\dot{x}\cos\theta, \ \frac{d^2z}{dt^2} = -g + 2\Omega\dot{x}\sin\theta.$$

Solve this order-by-order in $\Omega \ll 1$. The zero-th order solution is $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = h - \frac{1}{2}gt^2$. Plug these into the RHS of the above equation and then solve for the next order; leads to $x^{(1)} = \frac{1}{3}\Omega g t^3 \sin\theta$. So the object falls to $x > 0$ i.e to the East.

• Coriolis force leads to swirling cyclone air rotation around a low-pressure region, with the rotation vector pointing up in the Northern hemisphere (and down in Southern).

• Foucault Pendulum of length L. The EOM for the mass m bob is

$$m\frac{d^2\vec{r}}{dt^2} = \vec{T} + m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}, \qquad \vec{g} = \vec{g}_0 + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}.$$

Take the coordinate system as above, e.g. $\vec{\Omega} = \Omega(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \to \Omega(0, \sin\theta, \cos\theta)$ and for small displacements $(x, y) \ll L$ get $z \sim (x^2 + y^2)/L \approx 0$ and $T = mg$ and then

$$\frac{d^2}{dt^2}(x, y) \approx (-\frac{Tx}{mL} + 2\Omega\cos\theta\dot{y} - 2\Omega\dot{z}\sin\theta, -\frac{Ty}{mL} - 2\Omega\cos\theta\dot{x}) \to$$

$$\frac{d^2x}{dt^2} \approx -\omega_o^2 x + 2\dot{y}\Omega_z, \ \frac{d^2y}{dt^2} = -\omega_0^2 y - 2\dot{x}\Omega_z, \qquad \omega_0 \equiv \sqrt{g/L}, \quad \Omega_z \equiv \Omega\cos\theta.$$

Let $\eta(t) \equiv x(t) + iy(t)$ and then the EOM becomes $\frac{d^2\eta}{dt^2} = -\omega_0^2\eta - 2i\Omega_z\dot{\eta}$. The solutions are $\eta = C_1 e^{-i\alpha_+ t} + C_2 e^{-i\alpha_- t}$ where $\alpha_\pm = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2}$ are the roots of the characteristic equation and, since $\Omega \ll \omega_0$, we can approximate $\alpha_\pm \approx \Omega_z \pm \omega_0$. Take initial conditions

$x_0 = A$ and $y_0 = 0$. Then the solution is $\eta(t) = Ae^{-i\Omega_z t}\cos\omega_0 t$. At the North pole, it rotates through 360° in a day, which makes sense from the perspective of an inertial observer who sees the earth rotating and the pendulum staying in a plane (and it's opposite in the Southern hemisphere). For latitude around 42°, $\Omega_z \approx \frac{2}{3}\Omega \sim 240°/day$.

• Next topic (Section 10.1): the center of mass and rotation. Consider a collection of masses m_a, or a mass distribution $\rho(\vec{r})$. The total mass is $M = \sum_a m_a = \int dV \rho(\vec{r})$, where we use either a sum or an integral as appropriate, and we can convert between them via e.g. $\rho(\vec{r}) = \sum_a m_a \delta^3(\vec{r} - \vec{r}_a(t))$. The total momentum is $\vec{P} = \sum_a m_a \dot{\vec{r}}_a(t) = \int dV \rho(\vec{r}) \frac{d\vec{r}}{dt}$. Write $\vec{P} = M\dot{\vec{R}}$ where $\vec{R}$ is the center of mass (or center of momentum) position $\vec{R} \equiv \frac{1}{M}\sum_a m_a \vec{r}_a$ or $\vec{R} = \frac{1}{M}\int \vec{r} dm$, where $dm \equiv \rho(\vec{r})dV$. Using Newton's law, $\vec{F}_{ext} = M\frac{d^2\vec{R}}{dt^2}$.

Now take $\vec{r}_a = \vec{R} + \vec{r}'_a$. Here $\vec{r}_a$ is taken to be a vector in an inertial reference frame with a fixed origin. The angular momentum relative to that origin is $\vec{L} = \sum_a \vec{r}_a \times m_a \dot{\vec{r}}_a = \vec{R}\times\vec{P} + \sum_a \vec{r}'_a \times m_a \dot{\vec{r}}_a{}'$, where two terms drop out thanks to $\sum_a m_a \vec{r}'_a = 0$ and its derivative. This shows that the total angular momentum is that of the CM plus that relative to the CM. Now $\frac{d}{dt}\vec{R}\times\vec{P} = \vec{R}\times\dot{\vec{P}} = \vec{R}\times\vec{F}^{ext} = \vec{\Gamma}^{ext}$, the external torque acting on the CM. Likewise $\frac{d}{dt}\sum_a \vec{r}'_a \times \vec{p}_a = \sum_a \vec{r}'_a \times \vec{F}^{ext}_a = \vec{\Gamma}^{ext}|_{CM}$, the external torque relative to the CM.

The total kinetic energy is $T = \sum_a \frac{1}{2} m_a \dot{\vec{r}}_a{}^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a{}'^2$.

• For rotation around a fixed axis, we replace $\dot{\vec{r}}_a = \vec{\omega}\times\vec{r}_a$. Then $T_{rot} = \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a{}'^2 = \frac{1}{2}\sum_a m_a(\omega^2 r'^2_a - (\vec{\omega}\cdot\vec{r}'_a)^2) = \frac{1}{2}I_{jk}\omega_j\omega_k$ where $I_{jk} \equiv \sum_a m_a(\vec{r}^2_a \delta_{jk} - r_j r_k) = I_{kj}$ (so $I = I^T$).

• The moment of inertia tensor I_{jk} also enters in $L^{rot}_j = \sum_k I_{jk}\omega_k$, where $\vec{L}^{rot}$ is the CM rotational angular momentum: $\vec{L}^{rot} = \sum_a \vec{r}'_a \times m_a(\vec{\omega}\times\vec{r}'_a) = \sum_a m_a(\vec{\omega} r'^2_a - \vec{r}'_a(\vec{\omega}\cdot\vec{r}'_a))$.

E.g. take $\vec{\omega} = \omega\hat{z}$, then $\vec{v}_a = \omega\times\vec{r}_a = -\omega y_a\hat{x} + \omega x_a\hat{y}$ and $\ell_a = m_a\vec{r}_a\times\vec{v}_a = m_a\omega(-z_a x_a\hat{x} - z_a y_a\hat{y} + (x^2_a + y^2_a)\hat{z})$. The CM angular momentum thus has $\vec{L}_z = I_{zz}\omega$ where $I_{zz} \equiv \sum_a m_a\rho^2_a$, where $\rho^2_a = x^2_a + y^2_a$ is the distance of the point to the axis of rotation. The products of inertia enter in e.g. $L_x = I_{xz}\omega$ and $L_y = I_{yz}\omega$, with $I_{xz} = -\sum_a m_a x_a z_a$ and $I_{yz} = -\sum_a m_a y_a z_a$.

• Example: consider a wheel of radius R that is rolling without slipping with velocity $\vec{V} = V\hat{x}$. The center of the wheel has $y = R$, and we then find $\omega = -\omega\hat{z}$ with $\omega = V/R$. The velocity of a point on the wheel is $\vec{v} = \omega R\hat{x} + R\vec{\omega}\times\hat{r}$, where $\hat{r}$ points from the center of the wheel. For example, for the point of contact $\hat{r} = -\hat{y}$ and $\vec{v} = 0$, and for the top of the wheel $\vec{v} = 2\omega R\hat{x}$. A solid wheel has $L_z = I_{zz}\omega$, with $I_{zz} = \int dm\rho^2 = \frac{1}{2}MR^2$.

• Parallel axis theorem: replace $\vec{r}_a \to \vec{r}'_a = \vec{r}_a - \vec{d}$ (with $\sum_a m_a\vec{r}_a = 0$) for moment of inertia tensor for rotations about an axis displaced to $\vec{d}$. Get $I_{jk}(\vec{d}) = I_{jk}(0) + M(\vec{d}^2\delta_{jk} - d_j d_k)$. For example, for a solid wheel around a point on the rim this gives $I_{zz} = \frac{3}{2}MR^2$.