1/22/20 Lecture outline

⋆ **Reading: Taylor Chapter 9, Sections 10.1, 10.2**

• Last time: a general vector $\vec{Q}$ has $\frac{d\vec{Q}}{dt}|_{inertial} = \frac{d\vec{Q}}{dt}|_{prime} + \Omega \times \vec{Q}$, where *prime* denotes the rotating frame. This leads to a modified version of $\vec{F} = m\vec{a}$ In the prime (or "body") frame:

$$m\frac{d^2\vec{r}'}{dt^2} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}, \qquad \vec{F}_{cor} \equiv 2m\dot{\vec{r}}' \times \vec{\Omega}, \quad \vec{F}_{cf} = m\vec{\Omega} \times (\vec{r}' \times \vec{\Omega}).$$

The centrifugal force term can be evaluated using $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ to give $\vec{F}_{cf} = m\vec{r}'\Omega^2 - m\vec{\Omega}(\vec{r}' \cdot \vec{\Omega})$. For example, consider an observer on the surface of a rotating sphere of radius R (e.g. the earth), with $\vec{\Omega} = \Omega\hat{z}$, then $\vec{F}_{cf} = m\Omega^2 R(\sin\theta\cos\phi, \sin\theta\sin\phi, 0)$.

We can see $\frac{d^2\vec{r}'}{dt^2} = \frac{d^2\vec{r}}{dt^2} + \dot{\vec{r}}' \times \vec{\Omega} + \vec{\Omega} \times (\vec{r}' \times \vec{\Omega})$ also in spherical coordinates. Recall $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, and $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$, $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$. Then $d\hat{r} = \hat{\theta}d\theta + \sin\theta\hat{\phi}d\phi$, $d\hat{\theta} = -\hat{r}d\theta + \cos\theta\hat{\phi}d\phi$, $d\hat{\phi} = -\sin\theta\hat{r}d\phi - \cos\theta\hat{\theta}d\phi$. Writing out $\frac{d^2\vec{r}}{dt^2}$ in spherical coordinates, $\vec{r} = r\hat{r}$, the terms involving $\dot{\theta}^2$ and $\dot{\phi}^2$ can be interpreted in terms of centrifugal force and those involving $\dot{r}$ and $\dot{\phi}$ or $\dot{\theta}$ can be interpreted in terms of Coriolis force. As a special case (to shorten the formulae), consider $\theta = \pi/2$, where it is the same as for polar coordinates in the plane. Then $\vec{F} = m\frac{d^2\vec{r}}{dt^2} \to F_r = m(\frac{d^2r}{dt^2} - r\dot{\phi}^2)$ and $F_\phi = m(r\frac{d^2\phi}{dt^2} + 2\dot{r}\dot{\phi})$. If we re-write these as equations for $m\frac{d^2r}{dt^2}$ and $mr\frac{d^2\phi}{dt^2}$, the additional terms on the other side of the = sign are interpreted as $F_{r,cf} = mr^2\Omega^2$ and $F_{\phi,cor} = -2m\dot{r}\Omega$.

Example: free fall near the earth's surface (we omit writing the prime on $\vec{r}'$)

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}, \qquad \vec{g} = \vec{g}_0 + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

where $\vec{g}$ is the observed free-fall acceleration, which includes the centrifugal force term. Note that $\Omega^2_{earth}R_{earth} \approx 3.38 \times 10^{-2}ms^{-2}$ is about a 0.3% correction to g.

Take $\vec{r} \approx \vec{R}$ in $\vec{F}_{cf}$, which is a vector from the center of the earth to the position on the earth's surface where the experiment is done. Choose local coordinates near that location (assumed to be in the Northern hemisphere) such that $\hat{z}'$ points up (really it's $\hat{r}$) $\hat{y}'$ points North (it's really $-\hat{\theta}$), and $\hat{x}$ points East (it's really $\hat{\phi}$). In this coordinate system $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$. This gives

$$\frac{d^2x}{dt^2} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta), \ \frac{d^2y}{dt^2} = -2\Omega\dot{x}\cos\theta, \ \frac{d^2z}{dt^2} = -g + 2\Omega\dot{x}\sin\theta.$$

Solve this order-by-order in $\Omega \ll 1$. The zero-th order solution is $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = h - \frac{1}{2}gt^2$. Plug these into the RHS of the above equation and then solve for the next order; leads to $x^{(1)} = \frac{1}{3}\Omega g t^3 \sin\theta$. So the object falls to the East.