1/10/20 Lecture outline

$\star$ **Reading: Taylor sections 13.1, 13.2, 13.3 , Chapter 8.**

• Just for fun - some history: Copernican heliocentric theory with planets orbiting the sun 1543; controversial and defended by Galileo, 1633 trial. Tycho Brahe (1546-1601) took better data, and hired Kepler in 1600 as his assistant to help and to interpret the data. Found that circular orbits do not work; epicycles. Kepler's laws 1609-1619: (1) the planet's orbits are ellipses, with the sun at one of the two foci; (2) a line segment joining a planet to the sun sweeps out equal areas in equal times; (3) the square of the orbital period is directly proportional to the cube of the semi-major axis of the orbit, $\tau^2 \sim a^3$. The actual earth-sun distance was obtained (to 7% error) in 1672 by Giovanni Cassini by using parallax (to obtain the earth-mars distance). In 1672, this was used to determine the constancy of the speed of light by astronomer Olaf Roemer. In 1687 Newton showed how these follow from his universal law of gravitation and $\vec{F} = m\vec{a}$ and calculus.

• Last time: we reduced the 2-body motion problem $\to$ central force motion $\to$ an equivalent 1d problem with $L = \frac{1}{2}\mu\dot{r}^2 - U_{eff}(r)$ with $U_{eff} \equiv \frac{\ell^2}{2\mu r^2} + U(r)$. The conserved (CM) angular momentum is $\ell\hat{z}$ with $\ell = \mu r^2\dot{\phi}$ and the conserved energy is $H = E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$. We can then solve for the motion $\phi(t)$ and $r(t)$ by integration:

$$\phi(t) = \phi_0 + \int_0^t dt'\ell/\mu r^2(t'), \qquad t = \int_{r_0}^r \frac{dr}{\sqrt{\frac{2}{\mu}(E - U(r) - \frac{\ell^2}{2\mu r^2})}}.$$

which can be inverted to find $r(t)$. We thus have obtained $r(t)$ and $\phi(t)$.

Our main example: $U(r) = -Gm_1m_2/r$. $U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$. Illustrate turning points r_{min} and r_{max} for case $E < 0$: bounded orbit. For $E > 0$, there is a r_{min} but no r_{max}: unbounded orbit. Circular orbit at points $r = r_0$, where $U'_{eff}(r_0) = 0$, if $E = U_{eff}(r_0)$. Stable if $U''(r_0) > 0$.

• Conservation of angular momentum $\to$ equal areas swept out in equal times: the area element is $dA = \frac{1}{2}r^2 d\phi$ (compared with $dA = rdrd\phi$, we have done the $\int_0^{r(\phi)} rdr$). So $\dot{A} = \frac{1}{2}r^2\dot{\phi} = \ell/2\mu$ is a constant (Kepler's 2nd law). We can integrate this over a period τ for closed orbits: the orbit area is $A = \ell\tau/2\mu$.

• Mechanical similarity. Suppose that all lengths are rescaled: $\vec{r}_i \to \alpha\vec{r}_i$. Suppose that the potential energy is homogenous function of degree n, i.e. $U(\alpha\vec{r}_i) = \alpha^n U(\vec{r}_i)$. Examples: for $U = k/r$, we have $n = -1$; for $U = \frac{1}{2}kr^2$ we have $n = 2$. Suppose that we also scale time as $t \to \beta t$. Then velocities scale as $\vec{v} \to \frac{\alpha}{\beta}\vec{v}$, and kinetic energy scales as $T \to \frac{\alpha^2}{\beta^2} T$. Assuming that the potential scales homogeneously, the lagrangian also scales homogeneously if we take $\alpha^2/\beta^2 = \alpha^n$, i.e. $\beta = \alpha^{1-\frac{1}{2}n}$. Since the scale is just an overall factor, the equations of motion are unchanged. This is interesting: it implies that homogeneous potentials have similar solutions, differing only by rescalings, with properties simply related. Let a be a length scale in a solution, and a' be a length scale in the rescaled solution, with $a'/a = \alpha$. We then have

$$\frac{t'}{t} = \alpha^{1-\frac{1}{2}n}, \ \frac{v'}{v} = \alpha^{\frac{1}{2}n}, \ \frac{E'}{E} = \alpha^n, \ \vec{L}' = \vec{L}\alpha^{1+\frac{1}{2}n}.$$

For example, for $U = -k/r$, $n = -1$, and we immediately obtain that the period scales with the orbit size as $\tau \sim a^{1-\frac{1}{2}n} = a^{3/2}$. Recall also the viral theorem (HW exercise):

$$2\langle T\rangle = n\langle U\rangle.$$

Taking $E = T + U$ a constant, we have $E = \langle T\rangle + \langle U\rangle = (1 + \frac{1}{2}n)\langle U\rangle = (1 + \frac{2}{n})\langle T\rangle$.

In particular, for bounded orbits in a $1/r$ potential we have $\langle U\rangle = 2E$, $\langle T\rangle = -E$.

• Orbit equations have solution $r = r(t)$ and $\phi = \phi(t)$. Let's study the shape of the trajectory rather than the t dependence. Eliminating the parameter t, we can solve for $r = r(\phi)$. To do this, use

$$\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi},$$

where $u = 1/r$ is introduced for convenience. So

$$\frac{dr}{dt} = -\frac{\ell}{\mu}\frac{du}{d\phi}, \qquad \frac{d^2r}{dt^2} = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2},$$

and the r EOM becomes (with $F(r) = -dU/dr$)

$$u''(\phi) + u + \frac{\mu}{\ell^2 u^2}F(r) = 0.$$

Another option is to solve for $r(\phi)$ by using energy conservation at the outset. As usual, this is better because $F = ma$ gives a 2nd order differential equation, whereas energy conservation does one of those integrals for us, leaving just a first order equation remaining to integrate. Using the relation (see above) $\frac{d}{dt} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$, we get

$$\frac{dr}{dt} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi}$$

and substituting into energy conservation then gives

$$E = \tfrac{1}{2}\mu \left(\frac{\ell}{\mu r^2} \frac{dr}{d\phi} \right)^2 + U_{eff}(r),$$

which we can use to solve for $dr/d\phi$, and then integrate the equation to obtain

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{\ell dr/r^2}{\sqrt{2\mu(E - U_{eff}(r))}}$$

• Kepler orbits: $U(r) = -k/r$, so $F(r) = -k/r^2$. (Sign is chosen so that $k > 0$ corresponds to an attractive force). Get

$$u''(\phi) = -u(\phi) + k\mu/\ell^2,$$

which is like the free particle, if we substitute $w = u - k\mu/\ell^2$, so

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}, \qquad c \equiv \frac{\ell^2}{k\mu}. \tag{1}$$

where ϵ is a constant, which can be written in terms of the energy as

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

So $\epsilon < 1$ gives bounded orbits, and $\epsilon > 1$ gives unbounded orbits. For $\epsilon < 1$ the equation is an ellipse (with special case being a circle for $\epsilon = 0$). For $\epsilon > 1$ it is a hyperbola. For $\epsilon = 1$ it is a parabola.