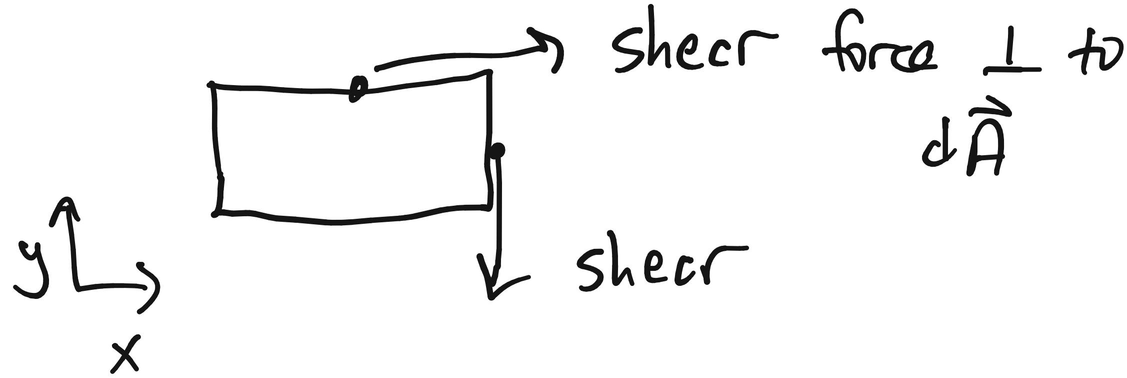
⋆ **Reading: Taylor chapter 16.1 to 16.11**

• Last time: recall pressure: in a static, ideal fluid, the surface force $d\vec{F}$ on any area element $d\vec{A}$ is $d\vec{F} = -pd\vec{A}$. More generally, the area element $d\vec{A}$ can have forces $dF^i = \sum_{j=1}^3 \sigma^{ij} dA^j$ where σ^{ij} is called the stress tensor and, for the case of a static, ideal fluid $\sigma^{ij} = -p\delta^{ij}$.

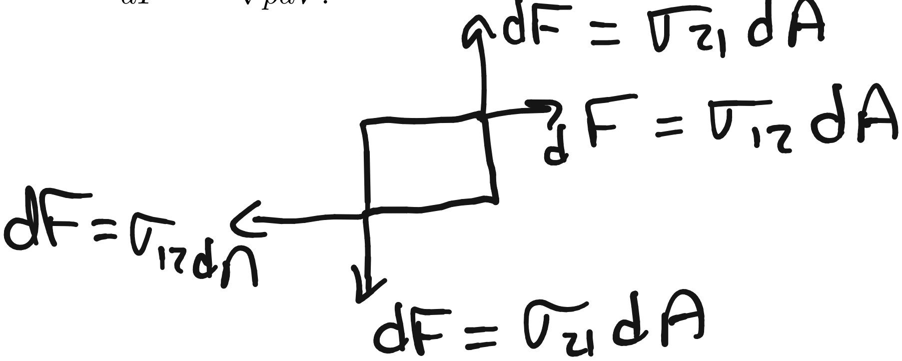
Ideal fluid $\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$

more generally, off diagonal components of $\sigma \longrightarrow$ shear forces

e.g.

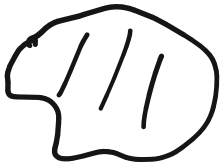


• If we consider a tiny square in the (12) plane then it would have torque around the 3 axis $\sim (\sigma^{12} - \sigma^{21})$ but if we scale the lengths to zero the angular momentum scales to zero more rapidly than this torque, which proves that $\sigma^{ij} = \sigma^{ji}$. The σ^{ij} stress tensor components are the space components of the stress-energy tensor $T^{\mu\nu}$ that we discussed in relativity: $T^{ij} = -\sigma^{ij}$. Indeed, $cP^i = \int_V d^3x T^{i0}$ and then $\frac{dP^i}{dt} = \int d^3x \partial_0 T^{i0} = -\int d^3x \partial_j T^{ij} = \int dA_j \sigma^{ij}$ where we used $\partial_\mu T^{\mu\nu} = 0$ and Gauss' law for integrating a divergence. The result fits with $dF^i = \sigma^{ij} dA^j$. For a closed surface ∂dV that is the boundary of dV, get $dF^i = \partial_j \sigma^{ij} dV$; for the case of an ideal static fluid this becomes $d\vec{F} = -\nabla p dV$.



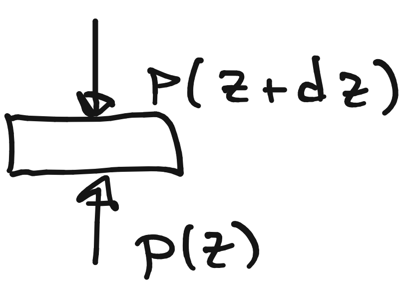
$$d\Gamma_3 = (\sigma_{21} - \sigma_{12})\, \ell\, dA \overset{!}{=} 0$$

$$\sigma^{ij} \overset{!}{=} \sigma^{ji}.$$



$$\frac{d\vec{P}^i}{dt} = \int_V dV\, \partial_j \sigma^{ij} = \int_{\partial V} dA^j \sigma^{ij}$$

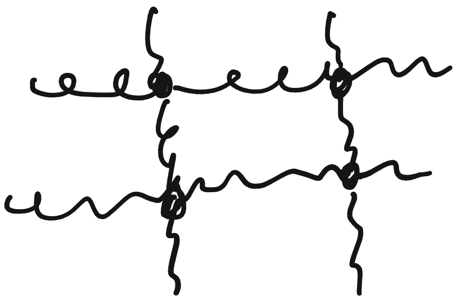
e.g.



$$dF_z = (p(z) - p(z+dz))\overset{dx\,dy}{dA}$$

$$= -\frac{\partial p}{\partial z}\, dx\,dy\,dz$$

• Consider displacements in a solid from equilibrium: $\vec{u}(t, \vec{x}) = \vec{x}' - \vec{x}$, where $\vec{x}'$ is the deformed position. The $\vec{u}$ is the analog of our displacement $y(x, t)$ in the case of a string. We can picture a bunch of coupled oscillators, and $\vec{u}(t, \vec{x})$ encodes their displacement from equilibrium. We expect to get a linear wave equation for $\vec{u}$ in the simplest cases, with small displacements from equilibrium.



$\vec{u}(t, \vec{x})$ = displaced position.

• The $\vec{u}$ lead to a 3×3 symmetric tensor called the strain tensor. One way to see it is to note that the deformation leads to $d\ell'^2 = d\vec{x}'^2 = (d\vec{x} + d\vec{u})^2 = d\ell^2 + 2u_{ij}dx_i dx_j$, where $u_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k) \approx \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, where the last term is dropped because displacements are usually small. u_{ij} is called the strain tensor. The book calls it **E**. As you checked in the HW, rotations do not contribute to u_{ij} because they are antisymmetric.

$$\vec{u} = \vec{x}' - \vec{x} \quad \to (d\vec{x}')^2 = (d\vec{x} + d\vec{u})^2$$

$$\equiv d\vec{x}^2 + 2U_{ij} dx_i dx_j$$

$$U_{ij} \approx \frac{1}{2}(\partial_i U_j + \partial_j U_i) = U_{ji}$$

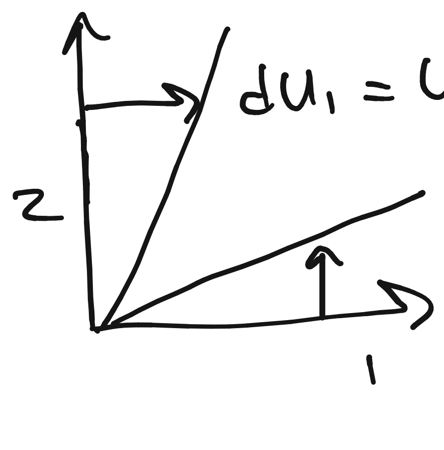
For a rotation $x'^i = R^i_j x^j$

get $U_{ij} = 0$, good: rotation $\neq$ strain.

e.g. $du^i = e\, dr^i \to U_{ij} = e\,\delta_{ij}$

stretches lengths by e, $\frac{dV}{V} = 3e$.

Or



$dU_1 = U_{12} dr_2$

shear

$dU_2 = U_{12} dr_1$

• So we have two 3×3 tensors: the stress tensor σ_{ij} related to the forces, and the strain tensor u_{ij} related to the displacements. For a small displacements, Hooke's law linearly relates forces to displacement, as in the case of a spring. More generally, it linearly relates σ_{ij} and u_{ij}:

$$\mathbf{u} = \frac{1}{3\alpha\beta}[3\alpha\sigma - (\alpha-\beta)\mathbf{1}(\mathrm{tr}\sigma)] \qquad \leftrightarrow \qquad \sigma = \frac{(\alpha-\beta)}{3\alpha}(\mathrm{tr}\sigma)\mathbf{1} + \beta\mathbf{u}.$$

Here $\alpha = 3MB$ and $\beta = 2SM$ where BM is the bulk modulus, and SM is the shear modulus. The bulk modulus arises as $dp = -BMdV/V$ for the case of pressure only, so $\sigma_{ij} = -p\delta_{ij}$ and then $u_{ij} = e\delta_{ij}$ so $e = -p/\alpha = \frac{1}{3}dV/V$. The shear modulus arises when $\mathrm{tr}u = 0$ and then $\sigma = \beta u$. Young's modulus is YM$= 3\alpha\beta/(2\alpha+\beta)$.

- The EOM for the displacement $\vec{u}$ is $\rho\frac{\partial^2 u^i}{\partial t^2} = \rho g^i + \partial_j \sigma^{ij}$. Using Hooke's law gives Navier's equation for $\vec{u}$: get $\partial_j \sigma^{ij} = (BM + \frac{1}{3}SM)\,\nabla^i(\nabla \cdot \vec{u}) + SM\,\nabla^2 u^i$ and thus

$$\rho\frac{\partial^2 \vec{u}}{\partial t^2} = \rho\vec{g} + (BM + \frac{1}{3}SM)\,\nabla(\nabla \cdot \vec{u}) + SM\,\nabla^2\vec{u}.$$

For longitudinal displacements e.g. $\vec{u} = (u_x(x,t), 0, 0)$, neglecting the $\vec{g}$ term, this gives a wave equation with $c_{long} = \sqrt{(BM + \frac{4}{3}SM)/\rho}$. For transverse displacements, e.g. $\vec{u} = (0, u_y(x,t), 0)$, this gives a wave equation with $c_{trans} = \sqrt{SM/\rho}$. Note that $c_{long} > c_{trans}$; gives a way to determine how far away the earthquake was.

longitudinal: $\rho \frac{\partial^2 u_x(x,t)}{\partial t^2} = \left(BM + \frac{4}{3}SM\right)\frac{\partial^2 u_x}{\partial x^2}$

transverse: $\rho \frac{\partial^2 u_y(x,t)}{\partial t^2} = SM \frac{\partial^2 u_y}{\partial x^2}$