3/11/20 Lecture outline

⋆ **Reading: Taylor chapter 16.1 to 16.11**

• Last time: string has $S = \int dtdx\mathcal{L}(\psi, \partial_t\psi, \partial_x\psi)$. Get Euler-Largange equations and also need to impose either $\delta\psi|_{end} = 0$ or $\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)})|_{end} = 0$; these are called Dirichlet (fixed end) and Neumann BCs, respectively. Let $\mathcal{P}^t \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}$ and $\mathcal{P}^x \equiv \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}$. Least action gives $\frac{\partial\mathcal{P}^t}{\partial t} + \frac{\partial\mathcal{P}^x}{\partial x} = \frac{\partial\mathcal{L}}{\partial\psi}$. Space and time translation symmetry leads to a conserved stress-energy tensor $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi - \eta_{\mu\nu}\mathcal{L}$, with $\partial_\mu T^{\mu\nu} = 0$. In particular, if $\mathcal{L}$ does not depend explicitly on t then $\mathcal{H} = T^{00}$ satisfies the conservation equation $\partial_t\mathcal{H} + \partial_x j_{\mathcal{E}} = 0$ with $j_{\mathcal{E}} = \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}\partial_t\psi$ the energy current flux.

Uniform string of mass density μ, tension T, with $\psi(t,x) = y(t,x)$ has $\mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 - \frac{1}{2}T(\partial_x y)^2$. The EOM are the wave equation $(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\psi(t,x) = 0$ with $c = \sqrt{T/\mu}$. Write it to look similar to relativity in 1+1d: $\mathcal{L} = \frac{1}{2}T\partial_\mu y\partial^\mu y$ where $\partial_\mu = (\frac{\partial}{c\partial t}, \frac{\partial}{\partial x})$ and $\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The EOM are $\partial_\mu\partial^\mu y = 0$. The stress-tensor is $T^{\mu\nu} = T\partial^\mu y\partial^\nu y - \frac{1}{2}T\eta^{\mu\nu}\partial_\lambda y\partial^\lambda y$, and we can verify that it is property symmetric and conserved $T^{\mu\nu} = T^{\nu\mu}$ and $\partial_\mu T^{\mu\nu} = 0$, and $T^{00} = \mathcal{H} = \frac{1}{2}\mu(\partial_t y)^2 + \frac{1}{2}T(\partial_x y)^2$, $j_{\mathcal{E}} = \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}\partial_t\psi = -T\partial_x y\partial_t y$.

• Consider string with $x \in [0, L]$. The BCs at the ends are either Dirichlet, $y(t,x)|_{end} = 0$, or Neumann, $\frac{\partial}{\partial x}y(t,x)|_{end} = 0$. Note that (N) BCs give $j_{\mathcal{E}} = 0$. Consider e.g. the case where the BCs are D at each end. Then $y(x,t) = \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{L})(A_n\cos\omega_n t + B_n\sin\omega_n t)$ solves the wave equation if $\omega_n = ck_n = cn\pi/L$. Get A_n and B_n from the FT of the initial position and velocity. Exercise: compute $\mathcal{H}$ and $j_{\mathcal{E}}$. Is the total energy $H = \int_0^L dx\mathcal{H}$ a constant? Consider also $j_{\mathcal{E}}$ at ends, and the time average.

• There is a natural generalization: $S_{KG} = \int dtd^3x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2)$. The EOM are the Klein-Gordon equation: $(\partial_\mu\partial^\mu + m^2)\phi = 0$. A solution is $\phi \sim a(k)e^{-ik_\mu x^\mu} + c.c.$ if $k^2 = m^2$. Using $p^\mu = \hbar k^\mu$ with $\hbar = 1$, this is the relativistic relation $p^2 = m^2$ for a particle of mass m. Indeed, upon quantization, ϕ is like the quantum $\hat{x}$ of a SHO, i.e. it can be written in terms of creation and annihilation operators, with quanta with mass m.

• Recall pressure: in a static, ideal fluid, the surface force $d\vec{F}$ on any area element $d\vec{A}$ is $d\vec{F} = -pd\vec{A}$. More generally, the area element $d\vec{A}$ can have forces $dF^i = \sum_{j=1}^3 \sigma^{ij}dA^j$ where σ^{ij} is called the stress tensor and, for the case of a static, ideal fluid $\sigma^{ij} = -p\delta^{ij}$. If we consider a tiny square in the (12) plane then it would have torque around the 3 axis $\sim (\sigma^{12} - \sigma^{21})$ but if we scale the lengths to zero the angular momentum scales to zero more rapidly than this torque, which proves that $\sigma^{ij} = \sigma^{ji}$. The σ^{ij} are the space components of the tensor $T^{\mu\nu}$ that we discussed in relativity.