3/9/20 Lecture outline

⋆ **Reading: Taylor chapter 16.1 to 16.11**

• Last time: string has $S=\int dtdx\mathcal{L}(\psi,\partial_t\psi,\partial_x\psi)$. Get Euler-Largange equations and also need to impose either $\delta\psi|_{end}=0$ or $\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)})|_{end}=0$; these are called Dirichlet (fixed end) and Neumann BCs, respectively. Let $\mathcal{P}^t\equiv\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}$ and $\mathcal{P}^x\equiv\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}$. Least action gives $\frac{\partial\mathcal{P}^t}{\partial t}+\frac{\partial\mathcal{P}^x}{\partial x}=\frac{\partial\mathcal{L}}{\partial\psi}$. The Hamiltonian is $H=\int dx\mathcal{H}$, where the Hamiltonian density is $\mathcal{H}=\mathcal{P}^t\partial_t\psi-\mathcal{L}$. As we will discuss, space and time translation symmetry leads to a conserved stress-energy tensor $T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi-\eta_{\mu\nu}\mathcal{L}$, with $\partial_\mu T^{\mu\nu}=0$. In particular, if $\mathcal{L}$ does not depend explicitly on t then $\mathcal{H}=T^{00}$ satisfies the conservation equation $\partial_t\mathcal{H}+\partial_x j_{\mathcal{E}}=0$ with $j_{\mathcal{E}}=\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}\partial_t\psi$ the energy current flux.

• Uniform string of mass density μ, tension T, with $\psi(t,x)=y(t,x)$ has $\mathcal{L}=\frac{1}{2}\mu(\partial_t y)^2-\frac{1}{2}T(\partial_x y)^2$. The EOM are the wave equation $(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2})\psi(t,x)=0$ with $c=\sqrt{T/\mu}$. The wave equation is solved by $y=y_R(x-ct)+y_L(x+ct)$ for arbitrary functions y_R and y_L.

The energy / Hamiltonian density is $\mathcal{H}=\mathcal{P}^t\partial_t\psi-\mathcal{L}=\frac{1}{2}\mu(\partial_t y)^2+\frac{1}{2}T(\partial_x y)^2$. To see its conservation law, note that $\partial_t\mathcal{H}+\partial_x(-T\partial_x y\partial_t y)=0$ so $j_{\mathcal{E}}=-T\partial_x y\partial_t y$ is the energy flux along the string. For $y=y_R(x-ct)+y_L(x+ct)$, get $\mathcal{E}=T[(y_R'(x-ct))^2+(y_L'(x+ct))^2]$ and $j_{\mathcal{E}}=cT[(y_R'(x-ct))^2-(y_L'(x+ct))^2]$.

The traveling wave also carries momentum flux density Π_x along the direction of the string. The energy and momentum conservation laws can be combined into a stress-energy tensor that looks similar to what we saw in relativity: $T^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial(\partial_\mu y)}\partial^\nu y-\eta^{\mu\nu}\mathcal{L}$ is the conservation law associated with translation symmetry in (ct,x), with $\partial_\mu T^{\mu\nu}=0$. We can write everything to look similar to relativity in 1+1d: $\mathcal{L}=\frac{1}{2}T\partial_\mu y\partial^\mu y$ where $\partial_\mu=(\frac{\partial}{c\partial t},\frac{\partial}{\partial x})$ and $\eta_{\mu\nu}=\eta^{\mu\nu}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The EOM are $\partial_\mu\partial^\mu y=0$. The stress-tensor is $T^{\mu\nu}=T\partial^\mu y\partial^\nu y-\frac{1}{2}T\eta^{\mu\nu}\partial_\lambda y\partial^\lambda y$, and we can verify that it is property symmetric and conserved $T^{\mu\nu}=T^{\nu\mu}$ and $\partial_\mu T^{\mu\nu}=0$, and $T^{00}=\mathcal{H}$, etc.