3/6/20 Lecture outline

⋆ **Reading: Taylor chapter 16.1 to 16.11**

• String: $S = \int dt dx \mathcal{L}(\psi, \partial_t\psi, \partial_x\psi)$ has $\delta S = \int dt dx \delta\psi(t,x)(\frac{\partial\mathcal{L}}{\partial\psi} - \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)} - \partial_x \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}) + \delta S_{bndy}$ where we integrated by parts and $\delta S_{bndy} = \int dt dx \partial_x(\delta\psi \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}) = \int dt \delta\psi \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}|_{ends}$ is the kind of term that is usually dropped (e.g. if the endpoints are at infinity and the fields anyway fall off there), but for a finite length string we need to impose separately that $S_{bndy} = 0$. There are two options: either $\delta\psi|_{end} = 0$ or $\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)})|_{end} = 0$; these are called Dirichlet (fixed end) and Neumann BCs, respectively.

• Let $\mathcal{P}^t \equiv \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}$ and $\mathcal{P}^x \equiv \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}$. Least action gives $\frac{\partial\mathcal{P}^t}{\partial t} + \frac{\partial\mathcal{P}^x}{\partial x} = \frac{\partial\mathcal{L}}{\partial\psi}$.

The Hamiltonian is $H = \int dx \mathcal{H}$, where the Hamiltonian density is $\mathcal{H} = \mathcal{P}^t\partial_t\psi - \mathcal{L}$. As we will discuss, space and time translation symmetry leads to a conserved stress-energy tensor $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi - \eta_{\mu\nu}\mathcal{L}$, with $\partial_\mu T^{\mu\nu} = 0$. In particular, if $\mathcal{L}$ does not depend explicitly on $t$ then $\mathcal{H} = T^{00}$ satisfies the conservation equation $\partial_t\mathcal{H} + \partial_x j_{\mathcal{E}} = 0$ with $j_{\mathcal{E}} = \frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}\partial_t\psi$ the energy current flux.

• Uniform string of mass density $\mu$, tension $T$, with $\psi(t,x) = y(t,x)$ the displacement from equilibrium in the $y$ direction. An element of length $dx$ has kinetic energy density $\frac{1}{2}\mu dx(\partial_t y)^2$ and potential energy density $T d\ell = \frac{1}{2}T(\frac{\partial y}{\partial x})^2 dx$ which comes from Taylor expanding $d\ell = \sqrt{dx^2 + dy^2} - dx$. Thus $S = \int dt dx \mathcal{L}$ with $\mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 - \frac{1}{2}T(\partial_x y)^2$. Varying $\delta_y S = 0$ gives the EOM, which can also be derived directly from $dF_y = \mu\partial_t^2 y = T\sin\phi_{x+dx} - T\sin\phi|_x$ and $\sin\phi \approx \tan\phi = \frac{\partial y}{\partial x}$ so $dF_y = dxT\frac{\partial^2\psi}{\partial x^2}$. The EOM are the wave equation $(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})\psi(t,x) = 0$ with $c = \sqrt{T/\mu}$. The wave equation is solved by $y = y_R(x-ct) + y_L(x+ct)$ for arbitrary functions $y_R$ and $y_L$.

The energy / Hamiltonian density is $\mathcal{H} = \mathcal{P}^t\partial_t\psi - \mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 + \frac{1}{2}T(\partial_x y)^2$. To see its conservation law, note that $\partial_t\mathcal{H} + \partial_x(-T\partial_x y\partial_t y) = 0$ so $j_{\mathcal{E}} = -T\partial_x y\partial_t y$ is the energy flux along the string. For $y = y_R(x-ct)+y_L(x+ct)$, get $\mathcal{E} = T[(y_R'(x-ct))^2+(y_L'(x+ct))^2]$ and $j_{\mathcal{E}} = cT[(y_R'(x-ct))^2 - (y_L'(x+ct))^2]$.