1/8/20 Lecture outline

⋆ **Reading: Taylor sections 13.1, 13.2, 13.3 , Chapter 8.**

• Briefly emphasize something from Hamiltonian mechanics: $H = H(q_a(t), p_a(t), t)$ so

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_a \left(\frac{\partial H}{\partial q_a}\dot{q}_a + \frac{\partial H}{\partial p_a}\dot{p}_a\right) = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Where we used Hamilton's equations $\dot{q}_a = \partial H/\partial p_a$ and $\dot{p}_a = -\partial H/\partial q_a$. So if H does not explicitly depend on time, then $dH/dt = 0$ and H is a constant of the motion, as discussed last time.

• Continue with two-body central force motion. The Lagrangian is assumed to be translationally invariant in space and time, and rotationally invariant, so $U(\vec{x}_1, \vec{x}_2, t) = U(r)$ with $r = |\vec{x}_1 - \vec{x}_2|$:

$$L = \tfrac{1}{2}m_1\dot{\vec{x}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{x}}_2^{\,2} - U(r).$$

The symmetries imply conservation of total momentum, energy, and angular momentum:

$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2, \quad H = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + U(r), \quad \vec{L}_{tot} = \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 \times \vec{p}_2$$

$$\dot{\vec{p}}_{tot} = \dot{H} = \dot{\vec{L}}_{tot} = 0.$$

• We can choose an inertial frame of reference where $\vec{p}_{tot} = 0$; this is called the center of momentum (or sometimes called center of mass) frame. This means that $\vec{R} = (m_1\vec{x}_1 + m_2\vec{x}_2)/M$, with $M \equiv m_1 + m_2$ is chosen to be a constant. The dynamical coordinate is then just the relative position $\vec{r} \equiv r_1 - \vec{r}_2$ and we can write

$$L = \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \to L = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(r), \qquad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}.$$

Then $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = \mu\dot{\vec{r}}$ and $\dot{\vec{p}} = -\nabla U(r) = -\frac{dU}{dr}\hat{r}$. The r here can be considered as in either spherical or cylindrical coordinates. Cylindrical coordinates are better: since $\vec{L}$ is constant, the motion stays in a plane. We can choose $\vec{L} = \ell\hat{z}$ and then the motion is in the (x, y) plane, $\dot{z} = 0$, and the motion has generalized coordinates r and ϕ with

$$L = \tfrac{1}{2}\mu\dot{r}^2 + \tfrac{1}{2}\mu r^2\dot{\phi}^2 - U(r),$$

and the EOM are

$$p_r = \mu\dot{r}, \quad \dot{p}_r = \frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{dU}{dr}, \quad p_\phi = \ell = \mu r^2\dot{\phi}, \quad \dot{p}_\phi = 0.$$

The ϕ EOM can be integrated to give

$$\phi(t) = \phi_0 + \int_0^t dt' \ell/\mu r^2(t').$$

The EOM for r is equivalent to a 1d theory with

$$L_{eff}(r, \dot{r}) = \tfrac{1}{2}\mu\dot{r}^2 - U_{eff}(r), \qquad U_{eff} \equiv \frac{\ell^2}{2mr^2} + U(r).$$

(Note that we substituted $\dot{\phi} = \ell/\mu r^2$ only *after* computing the r equations of motion, and then wrote U_{eff}. Eliminating $\dot{\phi}$ too soon gives a wrong sign term in U_{eff}.) Conservation of energy:

$$H = E = \tfrac{1}{2}\mu\dot{r}^2 + U_{eff}(r).$$

• Using above equations, we can solve the problem, reducing it to the computation of two integrals. Rewrite the energy conservation equation as

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}(E - U(r) - \frac{\ell^2}{2\mu r^2})}}$$

and integrate to get

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}(E - U(r) - \frac{\ell^2}{2\mu r^2})}},$$

which can be inverted to find $r(t)$. Then rewrite the conservation of angular momentum equation as

$$d\phi = \frac{\ell dt}{\mu r^2}$$

and integrate both sides to get

$$\phi - \phi_0 = \ell \int_0^t \frac{dt}{\mu r^2(t)}.$$

We thus have obtained, in principle, $r(t)$ and $\phi(t)$.

• The case $U \sim r^2$ is the 3d SHO, which separates into 3 copies of the 1d SHO. The case $U \sim 1/r$ is the Coulomb potential and it is also very special, e.g. it leads to closed orbits; this is related to the fact that it has an additional conserved quantity called the Laplace-Runge-Lenz vector $\vec{A} = \vec{p} \times \vec{L} - \mu k \hat{r}$ is conserved for $V = -k/r$.

Our main example: $U(r) = -Gm_1m_2/r$. $U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$. Illustrate turning points r_{min} and r_{max} for case $E < 0$: bounded orbit. For $E > 0$, there is a r_{min} but no r_{max}: unbounded orbit.