2/28/20 Lecture outline

⋆ **Reading: Taylor chapter 15.**

• Last time: electromagnetism is nicely relativistic. The source terms in Maxwell's equations organize into a 4-vector: $J^\mu = (c\rho, \vec{J})$. For a collection of charges, $J^\mu = \sum_a q_a \delta^3(\vec{x} - \vec{x}_a(t))(1, \dot{\vec{x}}_a(t))$.

Charge conservation of $Q = \int d^3x \rho$ and current conservation. It can be written as $\partial_\mu J^\mu = 0$, so the charge conservation requirement is Lorentz invariant: if it is satisfied in one frame, then it'll be satisfied in all.

• Note that $Q = \int d^3x \rho = \int d^3x' \rho'$ is Lorentz invariant, even though d^3x vs d^3x' has Lorentz contraction and ρ transforms as the time component of a 4-vector. Note that if $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$ then $d^4x' = \det(\frac{\partial x'}{\partial x}) d^4x$ and the Jacobian determinant is $\det \Lambda = 1$, so $d^4x = d^4x'$ is Lorentz invariant. Since dx^0 transforms the same as ρ, as the time component of a 4-vector, this shows that $d^3x\rho = d^3x'\rho'$.

• The electric and magnetic fields combine into $F^{\mu\nu}$ as $F^{i0} = E^i$ and $F^{ij} = -\epsilon^{ijk} B_k$. Under a Lorentz transformation, 4-vectors have $a^\mu = \Lambda^\mu_{\nu'} a^{\nu'}$, and $F^{\mu\nu}$ behaves like that for each index, i.e. $F = \Lambda^T F' \Lambda$. Taking the two frames to have relative velocity v_{rel} along the $\widehat{x}$ axis, this implies that (setting $c = 1$): $E_x = E'_x$, $B_x = B'_x$, and

$$\begin{pmatrix} E_y \\ B_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E'_y \\ B'_z \end{pmatrix}, \quad \begin{pmatrix} E_z \\ B_y \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E'_z \\ B'_y \end{pmatrix}.$$

• The Lorentz force law can be written as $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$ which we can write as $\vec{f} = \frac{q}{c}(u^0 \vec{E} + \vec{u} \times \vec{B})$ and the equation for power gives $\frac{dp^0}{d\tau} = \frac{q}{c} \vec{u} \cdot \vec{E}$ so they can be written together as $\frac{dp^\mu}{d\tau} = f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$, so they transform relativistically.

• Maxwell's equations can be written as $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu$ and $\partial_\mu \tilde{F}^{\mu\nu} = 0$, where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The second equations (absence of magnetic charges) can be solved via $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.