2/24/20 Lecture outline

⋆ **Reading: Taylor chapter 15.**

• Last time: dx^μ is a 4-vector, and $d\tau = dt/\gamma$ is a Lorentz scalar, $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(c, \vec{v})$ is a 4-vector version of velocity. Note that $u_\mu u^\mu = c^2$.

• Energy and momentum combine into a 4-vector $p^\mu = (E/c, \vec{p})$, with $p_\mu p^\mu = (mc)^2$.

So the mass m is Lorentz invariant. The energy is $E = \sqrt{(cp)^2 + (mc^2)^2}$ which, for $cp \ll mc^2$ we can expand as $E \approx mc^2 + \frac{p^2}{2m} + \ldots$.

For massive particles, $p^\mu = mu^\mu$, i.e. $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$. For a $m = 0$ massless particle, like a photon, $p^\mu p_\mu = 0$ but we can still define E and $\vec{p}$. In fact, $p^\mu = \hbar k^\mu$ where $k^\mu = (\omega/c, \vec{k})$ with $\omega = ck$. For both massive and massless particles, $\vec{v} = \vec{p}c^2/E$.

• 4-vector version of acceleration: $a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}u^\mu = \gamma\frac{d}{dt}(\gamma\frac{dx^\mu}{dt}) = \gamma^2\frac{d^2x^\mu}{dt^2} + \gamma\frac{dx^\mu}{dt}\frac{d\gamma}{dt}$. The space component of the first term is proportional to the non-relativistic acceleration, but the vector in the second term need not even point in the same direction.

• The 4-vector version of force is $f^\mu = (power/c, \vec{f})$ and Newton's laws are $f^\mu = \frac{dp^\mu}{d\tau}$. Note that $\frac{dE}{d\tau} = \frac{d}{d\tau}\sqrt{(c\vec{p})^2 + (mc^2)^2} = \frac{c^2}{E}\vec{p}\cdot\frac{d\vec{p}}{d\tau} = \vec{v}\cdot\vec{f}$.

• Conservation of energy and momentum: for an isolated system we have translation invariance, and $f^\mu_{tot,ext} = 0$ and then $\sum_{initial particles,i} p^\mu_i = \sum_{final particles,f} p^\mu_f$. For example, $n \to p^+e^-\bar{\nu}_e$, or $\pi^0 \to \gamma\gamma$. Mass is not conserved (e.g. π^0 is massive and the photons γ are massless), but the total energy and momentum are conserved. E.g. in the CM frame $\pi^0 \to \gamma\gamma$ each photon has $E = c|\vec{p}| = \frac{1}{2}m_{\pi^0}c^2$. Write $p^\mu_1 = p^\mu_2 + p^\mu_3$ and illustrate using $p^2 = m^2$ etc.

• Lorentz transformations are a symmetry of Nature, and this is ensured by having the action be Lorentz invariant. Let's start with a free, massive particle; we want to generalize $S \approx \int dt(\frac{1}{2}m\vec{v}^2)$. To get a Lorentz invariant, we can take $S = -mc^2\int_{worldline} d\tau = -mc^2\int dt\sqrt{1 - \vec{v}^2/c^2}$. Gives $\vec{p} = \frac{\partial L}{\partial \vec{v}} = \gamma m\vec{v}$ and properly reduces to $L \approx \frac{1}{2}m\vec{v}^2$ if $v \ll c$.