2/12/20 Lecture outline

⋆ **Reading: Taylor sections 15.1, 15.2, 15.3, 15.10.**

• Last time "boost symmetry:" frames with constant relative velocity are all physically equivalent. For low relative velocity, the relation is approximately given by the Galilean transformation $x' \approx x - vt$, $t' \approx t$, $y' = y$, $z' = z$. But this does not preserve the wave equation for $\vec{E}$ and $\vec{B}$: it has $\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \nabla'^2) \neq \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)$ and would make the wrong, boost-breaking prediction that light moves at speeds $c \pm v$ in the prime frame.

The correct transformation preserves $ds^2 = (cdt)^2 - d\vec{x}^2$ (it also preserves $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$). Since the two frames are related by $v \to -v$, this is the only possibility that ensures $ds^2 = 0 \leftrightarrow ds'^2 = 0$. Argue also that $dy' = dy$ and $dz' = dz$ for relative motion along the x-axis. So the transformation has $(cdt)^2 - dx^2 = (cdt')^2 - dx'^2$. If there were a relative + sign, it would be a rotation matrix and the dot product would be preserved using $\sin^2\theta + \cos^2\theta = 1$. The relative minus sign leads to a close cousin: $\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$ where we can exhibit the analogy via $\gamma \equiv \cosh\phi$, $\beta\gamma \equiv \sinh\phi$ and the matrix has unit determinant because $\gamma = 1/\sqrt{1-\beta^2}$.

The Lorentz transformation is $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ with $\beta = v/c$, and the Galilean transformation is recovered in the $\beta \ll 1$ limit.

The $\phi = \tanh^{-1}(v/c)$ above is called the rapidity, and it has the nice property that a Lorentz transformation parameterized by ϕ_1, combined with another parameterized by ϕ_2, is equivalent to a Lorentz transformation parameterized by $\phi_1 + \phi_2$. So rapidities simply add, while the addition of velocities formula is more complicated, (to be discussed in more detail soon).

The Lorentz transformation preserves the wave equation $\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \nabla'^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$, so the two observers agree that light waves travel at $v = c$ in all Lorentz-related frames.

The inverse Lorentz transformation is related by $v \to -v$; makes sense.

• A heartbeat in the $'$ frame has $dx' = 0$ and this in the Lorentz transformation leads to $dt = \gamma dt'$: time dilation. This can also be illustrated by considering a light flash in the $'$ frame that is emitted at the origin of spacetime, goes up the y' axis, gets reflected off a mirror at $y' = h$, and gets recorded by a detector at $(ct', x', y', z') = (\Delta t', 0, 0, 0)$. The time between the events in the lab frame is found from the Lorentz transformation to be $\Delta t = \gamma \Delta t'$: time dilation. Note that the two events have $\Delta s^2 = (c\Delta t')^2$ in the prime frame and $\Delta s^2 = (c\Delta t)^2 - \Delta x^2$ and they agree since $\Delta t = \gamma \Delta t'$ and $\Delta x^2 = v^2 \Delta t^2$.