1/6/20 Lecture outline

⋆ **Reading: Taylor sections 13.1, 13.2, 13.3, Chapter 8.**

• Recall classical mechanics v1: $\vec{F} = \dot{\vec{p}}$, with $\vec{p} = m\dot{\vec{x}}$; 2nd order ODE for $\vec{x}(t)$.

Classical mechanics v2: Least action $S = \int dt L(q_a, \dot{q}_a, t)$, $\delta S = 0 \to$ Euler Lagrange equations for generalized coordinates and momenta: $\dot{p}_a = \frac{\partial L}{\partial q_a}$, with $p_a = \frac{\partial L}{\partial \dot{q}_a}$. Can focus on the right coordinate, e.g. the angle for a pendulum. Symmetries $\leftrightarrow$ conservation laws (Noether): translation invariance $\leftrightarrow$ conservation of momentum, rotational symmetry $\leftrightarrow$ conservation of angular momentum, time translation invariance $\leftrightarrow$ conservation of energy.

• Classical mechanics v3: Hamilton's description. The Hamilton is related to the Lagrangian by a Legendre transform (similar transforms appear in thermodynamics)

$$H(q_a, p_a, t) \equiv \sum_a p_a \dot{q}_a - L(q_a, \dot{q}_a, t), \qquad L(q_a, \dot{q}_a, t) \equiv \sum_a p_a \dot{q}_a - H(q_a, p_a, t).$$

The Lagrangian depends on the velocities $\dot{q}_a$, whereas the Hamilton is expressed instead in terms of the momenta p_a. To see what H depends on, note that, using the EL equations,

$$dH = \sum_a \left(dp_a \dot{q}_a + p_a d\dot{q}_a - \frac{\partial L}{\partial q_a} dq_a - \frac{\partial L}{\partial \dot{q}_a} d\dot{q}_a \right) - \frac{\partial L}{\partial t} dt = \sum_a \left(dp_a \dot{q}_a - \dot{p}_a dq_a \right) - \frac{\partial L}{\partial t} dt$$

the cancellation of the $d\dot{q}$ term shows that H should not be regarded as depending on $\dot{q}$. Moreover, we can read off from the above **Hamilton's equations**

$$\dot{q}_a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q_a}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_a \left(\frac{\partial H}{\partial q_a} \dot{q}_a + \frac{\partial H}{\partial p_a} \dot{p}_a \right) = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

The (q_a, p_a) variables are called **phase space** and the second order ODE for $q_a(t)$ is replaced with two first order ODEs for $q_a(t)$ and $p_a(t)$.

• Example: the SHO, with $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2$. The EL equations are $\frac{d^2 x}{dt^2} = -\omega^2 x$, and are solved by $x = A\cos(\omega t + \varphi)$, with A and φ the expected two constants of integration, which can be determined by the initial position and velocity. The Hamiltonian is $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$ and Hamilton's equations are $\dot{x} = p/m$ and $\dot{p} = -m\omega^2 x$. The solution of these equations is an ellipse in phase space $x = A\cos(\omega t + \varphi)$, $p = m\dot{x} = -m\omega A \sin(\omega t + \varphi)$. Since H does not depend explicitly on t, the Hamiltonian is a constant of the motion, and in this case this gives an ellipse:

$$H(x, p) = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2 = \tfrac{1}{2} m\omega^2 A^2 = \text{constant}.$$

• The 110a class last quarter did not get to two-body central force motion. This is an important topic, so we will cover it now.

Consider two point masses, m_1 and m_2, with locations $\vec{x}_1(t)$ and $\vec{x}_2(t)$. We can apply this for example, to the sun and the earth in the approximation where we ignore the fact that they're not really point masses; this is a pretty good approximation because their separation is so large compared to their radii. The Lagrangian is assumed to be translationally invariant in space and time, and rotationally invariant, so $U(\vec{x}_1, \vec{x}_2, t) = U(r)$ with $r = |\vec{x}_1 - \vec{x}_2|$:

$$L = \tfrac{1}{2}m_1\dot{\vec{x}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{x}}_2^{\,2} - U(r).$$

The symmetries imply conservation of total momentum, energy, and angular momentum:

$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2, \quad H = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} + U(r), \quad \vec{L}_{tot} = \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 \times \vec{p}_2$$

$$\dot{\vec{p}}_{tot} = \dot{H} = \dot{\vec{L}}_{tot} = 0.$$

We can choose an inertial frame of reference where $\vec{p}_{tot} = 0$; this is called the center of momentum (or sometimes called center of mass) frame. This means that $\vec{R} = (m_1\vec{x}_1 + m_2\vec{x}_2)/M$, with $M \equiv m_1 + m_2$ is chosen to be a constant. The dynamical coordinate is then just the relative position $\vec{r} \equiv r_1 - \vec{r}_2$ and we can write

$$L = \tfrac{1}{2}M\dot{\vec{R}}^{2} + \tfrac{1}{2}\mu\dot{\vec{r}}^{2} - U(r) \to L = \tfrac{1}{2}\mu\dot{\vec{r}}^{2} - U(r), \qquad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}.$$

Then $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = \mu\dot{\vec{r}}$ and $\dot{\vec{p}} = -\nabla U(r) = -\frac{dU}{dr}\widehat{r}$.