140a Lecture 18, 3/12/19

⋆ Week 10 reading: Blundell+Blundell, chapters 23, 24, 25, 28.1, 28.2, 28.3.

• Continue where we left off last time, considering a thermal collection of photons in a box. We saw that the wave counting gives $2Vd^3\vec{k}/(2\pi)^3$ (where the 2 is for polarizations) and $\omega = ck$, photons have energy $E = \hbar\omega$ and $\vec{p} = \hbar k$, with $\omega = ck$. The energy density of a gas of photons of frequency ω is $u = U/V = n\hbar\omega$ where $n = N/V$ is the photon density. The pressure of a gas of photons is $p = u/3$. The number of photons hitting a unit area of the container wall per second is $\Phi = \frac{1}{4}nc$ and thus the power incident on the wall per unit area is $P = \hbar\omega\Phi = \frac{1}{4}uc$. The thermodynamic relations $(\partial U/\partial V)_T = T(\partial S/\partial V)_T - p = T(\partial p/\partial T)_V - p$ becomes $u = \frac{1}{3}(T(\partial u/\partial T)_V - u)$ which leads to $4dT/T = du/u$ and thus $P = \frac{1}{4}uc = \sigma T^4$. Let's now show how to get this, and derive the value of the Stefan-Boltzmann constant σ.

Each Fourier mode of the light in the box is like a SHO, and there is a factor of two from the two polarizations. As we discussed earlier, the number of modes in a box of volume V is $V\frac{d^3k}{(2\pi)^3}$. The total internal energy is then

$$U = 2V\int \frac{d^3k}{(2\pi)^3}\hbar ck(\tfrac{1}{2} + \frac{1}{e^{\beta\hbar\omega}-1}) \to \frac{4V\sigma}{c}T^4 \qquad \text{with} \quad \sigma = \frac{\pi^2 k_B^4}{60c^2\hbar^3}.$$

The $\frac{1}{2}$ is the vacuum energy of the SHO, which we won't worry about here (mention briefly the cosmological constant). The blackbody distribution $u(\omega) = \hbar\omega^3/\pi^2c^3(e^{\beta\hbar\omega}-1)$ reproduces the classical equipartition theorem answer for $\beta\hbar\omega \ll 1$, and the exponential in the denominator cures the ultraviolet catastrophe of the classical equipartition theorem for large ω: recall from your QM classes that this was how Planck first introduced $\hbar$ and he wrote down the answer for $u(\omega)$ by fitting, without understanding that light comes in quantized photons. That understanding came later, from Einstein who first wrote down $E = \hbar\omega$ to explain both this and especially the photoelectric effect.

The cosmic microwave background radiation is the afterglow from the early universe, and is a blackbody spectrum with temperature $T \approx 3K$ (with tiny temperature anisotropies measured in different parts of the sky by cosmology experiments).

• The above description was in terms of the canonical ensemble for the SHO energy levels. Alternatively and equivalently, we can get it from the grand canonical ensemble for occupation number n of the energy $E = \hbar\omega$. We saw before that identical bosons in this description have $\langle n\rangle = (e^{\beta(E-\mu)}-1)^{-1}$, and this matches the above if we set $\mu = 0$. In terms of the microcanonical ensemble, we saw that μ is related to the Lagrange multiplier that enforces the particle number conservation law $\sum_i n_i = N$, but there is no such constraint on the number of photons, which is why we can set $\mu = 0$. Recall that $G = F + pV = \mu N$, so $\mu = 0$ gives $F = -pV = -k_B T \ln Z = -k_B T 2V(4\pi)(2\pi\hbar)^{-3} \int_0^\infty p^2 dp \ln(1 - e^{-cp/k_B T}) = -U/3$ where $p^2 dp = d(p^3/3)$ was integrated by parts to get the same integral for U as above. This gives yet another way to see the 1/3 factor that was obtained in several ways last lecture. Also, $S = (U - F)/T = \frac{4}{3}(U/T) \propto VT^3$ and $C_V = T(\frac{\partial S}{\partial T})_V = 3S$.

• Continue along these lines for relativistic gases. $E = \sqrt{c^2 p^2 + (mc^2)^2}$. In the ultrarelativistic limit, $E \approx cp$. The single particle partition function is then

$$Z_1 = V \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} e^{-\beta cp} = \frac{V}{2\pi^2} (\frac{k_B T}{\hbar c})^3 \int_0^\infty e^{-x} x^2 dx = \frac{V}{\pi^2} (\frac{k_B T}{\hbar c})^3.$$

Recall that in the non-relativistic case we found $Z_1 \propto VT^{3/2}$. Write $Z_1 = V/\Lambda^3$ in the relativistic case, with $\Lambda \sim 1/T$.

For low-density the partition function for N indistinguishable relativistic particles is $Z_N = Z_1^N/N!$ and thus $\ln Z_N \approx N \ln V + 3N \ln T + const$. So $U = -\frac{d}{d\beta} \ln Z = 3Nk_B T$ (vs $\frac{3}{2} N k_B T$ in the non-relativistic case) and $C_V = 3Nk_B$ and $F = -k_B T \ln Z_N$ gives $p = -(\frac{\partial F}{\partial V})_T = Nk_B T/V$ (same ideal gas law). So we again get $p = u/3$ with $u = U/V$. Find the entropy $S = (U - T)/T = Nk_B(4 - \ln(n\Lambda^3))$.