140a Lecture 16, 3/5/19

⋆ Week 9 reading: Blundell+Blundell, chapters 29, 23, 24

• Continue where we left off last time. Grand partition function: imagine that our small system, of energy ϵ and particle number N, is in contact with a huge reservoir with internal energy $U-\epsilon$ and particle number $\mathcal{N}-N$, and consider the number of states of the combined system plus reservoir, coming from the number of states in the reservoir. Taylor expand $S(U-\epsilon,\mathcal{N}-N)\approx S(U,\mathcal{N})-(\epsilon-\mu N)/T$ and thus $P(\epsilon,N)\propto e^{S(U-\epsilon,\mathcal{N}-N)/k_B}\propto e^{\beta(\mu N-\epsilon)}$. So define the grand partition function as $\mathcal{Z}(T,\mu,V)\equiv\sum_i e^{\beta(\mu N_i-E_i)}$. Get $N=k_BT(\partial\ln\mathcal{Z}/\partial\mu)_{\beta,V}$. $U=-(\partial\ln\mathcal{Z}/\partial\beta)_{\mu,V}+\mu N$. $S=-k_B\sum_i P_i\ln P_i=T^{-1}(U-\mu N+k_BT\ln\mathcal{Z})$. Taking $\Phi_G\equiv -k_BT\ln\mathcal{Z}$, show that $\Phi_G=F-\mu N$ and $S=-(\partial\Phi_G)/\partial T)_{V,\mu}$, $p=-(\partial\Phi_G/\partial V)_{T,\mu}$, $N=-(\partial\Phi_G/\partial\mu)_{T,V}$.

• Next topic: Bose-Einstein and Fermi-Dirac distributions. Recall from your QM class that the wavefunction for two identical particles satisfies $\psi(\vec{r}_1,\vec{r}_2)=\pm\phi(\vec{r}_2,\vec{r}_1)$, where it is a $+$ sign for identical bosons (e.g. photons) and a $-$ sign for identical fermions (e.g. electrons). Consider e.g. a 2-state system. For two distinguishable particles (Fred and Jeremiah were the names chosen in class) there are four possible states which we can write as $|0\rangle_F|0\rangle_J$, $|0\rangle_F|1\rangle_J$, $|1\rangle_F|0\rangle_J$, $|1\rangle_F|1\rangle_J$. If they are indistinguishable, we drop the name tags and classically we have the three states $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|1\rangle$. For indistinguishable bosons there are indeed three states but the wavefunctions are actually $|0\rangle|0\rangle$, $\frac{1}{\sqrt{2}}(|0\rangle|1\rangle+|1\rangle|0\rangle)$, $|1\rangle|1\rangle$. For indistinguishable fermions there is actually only one state, with wavefunction $\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle)$.

• Consider e.g. a 1-state system, where the one state has energy E. Consider the grand partition function $\mathcal{Z}=\sum_n e^{n\beta(\mu-E)}$, where n is the occupation number. Then $\langle n\rangle=-k_BT\partial\ln\mathcal{Z}/\partial E$.

For Fermions, the only allowed values in $\sum_n$ are $n=0,1$, so $\mathcal{Z}_{fermions}=1+e^{\beta(\mu-E)}$. For bosons we sum the geometric series over all $n=0,\dots\infty$ to get $\mathcal{Z}_{bosons}=(1-e^{-\beta(\mu-E)})^{-1}$. Both can be written as giving $\ln\mathcal{Z}_{F,B}=\pm\ln(1\pm e^{\beta(\mu-E)})$. Get $\langle n\rangle=(e^{\beta(E-\mu)}\pm 1)^{-1}$. The distribution function is the mean occupation of a single particle state with energy E and is given by $f_{FD}(E)=(e^{\beta(E-\mu)}+1)^{-1}$ and $f_{BE}(E)=(e^{\beta(E-\mu)}-1)^{-1}$.