Physics 212b, Ken Intriligator lecture 9, Feb 7, 2018

• Aside: in QFT, the fields form representations of the Lorentz group, which can be thought of (in Euclidean space) as $SO(4) \cong SU(2)_L \times SU(2)_R$, i.e. the same group that we met for the 3d Coulomb potential. Fermions can be left-handed chiral, in the $(2,1)$ dimensional representation or right-handed chiral in the $(1,2)$ representation, if they are massless. All of the Fermions of the standard model are massless, and chiral, in the needed sense; they get their mass from coupling to a scalar field, called the Higgs field, which has a Bose-Einstein condensate. Parity exchanges left-handed with right-handed.

It is useful to also consider charge conjugation C, which e.g. takes $CA_\mu C = -A_\mu$ and replaces e.g. the electron field with its conjugate positron field. It can be shown that the combination CPT can only be violated if Lorentz invariance is violated; there is no current reason to expect such violation.

Most of the Standard Model preserves C, P, and T. The weak interactions $SU(2)_W$ violate P and C separately, preserving CP and T; for example, P exchanges $SU(2)_L \leftrightarrow SU(2)_R$ if we think about rotations + boosts in terms of $SU(2)_L \times SU(2)_R$ (similar to what we saw for the hydrogen atom's symmetry, with $SU(2)_I$ and $SU(2)_K$), but the weak interactions only couple to the $SU(2)_L$ part of the fermions. This is the fundamental reason for the parity violating effect observed first by Chien-Shiung Wu in 1956: $Co \to Ni + e^- + \bar{\nu}^e + 2\gamma$ (i.e. $n \to p + e^- + \bar{\nu}^e$, i.e. $d \to u + W^-$ and $W^- \to e^- + \bar{\nu}_e$) and Wu's experiment showed that the electrons coming out prefer to be anti-aligned with the nuclear spin $\to$ Lee + Yang Nobel prize in 1957.

Feynman's story. Titanic movie set.

There is a $\theta \vec{E}^a \cdot \vec{B}^a$ interaction for the strong force, that violates CP and T separately. The particles come in 3 families and thanks to the 3rd family there are couplings of Fermions to the Higgs field that violate CP and T, preserving CPT; they violate T because they have a complex phase that cannot be rotated away, and the fact that $\mathcal{L}$ has a non-real coefficient violates T because it is an anti-unitary operator. CP violation is needed to explain the observed matter / anti-matter asymmetry of the Universe, but the CP violation in the SM is quite tiny and seems to be insufficient to get the observed asymmetry; this is one of many hints about needed physics beyond the SM.

• Permutation symmetry. E.g. 2 particle state for free, non-interacting particles are given by $|k, k'\rangle = |k\rangle_1 \otimes |k'\rangle_2$. Then $P_{12}|k, k'\rangle = |k', k\rangle$. Note that $P_{12}^2 = 1$, so the eigenvalues are ± 1. Can form $|k, k'\rangle_\pm \equiv \frac{1}{\sqrt{2}}(|k, k'\rangle \pm |k', k\rangle)$, which are eigenstates of P_{12}

with eigenvalue ± 1. The exchange generator P_{12} is the symmetry generator of the group $Z_2 \cong S_2$, and it can be represented by either 1 or -1.

Now consider 2 particles with $H = H_1 + H_2 + H_{int}$, where H_{int} is symmetric under exchange of the two particles and $H_1(\vec{x}_1, \vec{p}_1) = H_2(\vec{x}_2, \vec{p}_2)$. Then $P_{12}H = HP_{12}$ where P_{12} is the exchange operator. Can take states to be eigenkets of P_{12}, with eigenvalue ± 1.

For 3 objects there are two groups that we can consider: Z_3 is the group of all cyclic elements (3 elements) , and S_3 is the group of arbitrary permutations (3!=6 elements). For 3 decoupled 1d SHOs, for example, we can consider $|n_1, n_2, n_3\rangle \equiv |n_1\rangle \otimes |n_2\rangle \otimes |n_2\rangle$ (this is the same as for the 3d SHO, but the interpretation here is different). Then we can consider $|n_1, n_2, n_3\rangle_\pm \equiv \frac{1}{\sqrt{6}}(|n_1, n_2, n_3\rangle \pm (perms))$. All permutations can be written as products of exchanges, e.g. cyclic permutations are an even number of exchanges. The signs in the previous wave function is $+1$ for cyclic permutations of (n_1, n_2, n_3) and -1 for the others, like ϵ_{ijk}.

• Identical particles: if there are N identical bosons, the wave function must be in a fully symmetric representation of S_N, i.e. $P_{ij} = 1$ for exchanging any two particles. For N identical fermions, must be eigenstates with eigenvalue -1 for all P_{ij}. Spin statistics theorem: half integer spin particles are fermions, and integer spin particles are bosons.

• Two electron system $\psi(1,2) = \phi(\vec{x}_1, \vec{x}_2)\chi$, where $\chi = |j = 1, m\rangle_S$ or $|j = 0, m = 0\rangle_A$. The condition $\psi(2,1) = -\psi(1,2)$ implies that either ϕ has $P_{12} = +1$ and $\chi = \chi_A$ (singlet), or ϕ has $P_{12} = -1$ and $\chi = \chi_S$ (triplet); so the electrons must avoid each other in the triplet state (while for the singlet there is enhanced probability of finding them nearby).

Show $\vec{S}_1 \cdot \vec{S}_2 = \hbar^2/4$ for the triplet and $\vec{S}_1 \cdot \vec{S}_2 = -3\hbar^2/4$ for the singlet.

• Consider two distant electrons, with $\psi(1,2) = \frac{1}{\sqrt{2}}(\psi_1(\vec{r}_1)\psi_2(\vec{r}_2) \pm (\vec{r}_1 \leftrightarrow \vec{r}_2))$. If we don't observe electron 2, then $P(\vec{r}) = \int d^3\vec{r}_2 |\psi(\vec{r}, \vec{r}_2)|^2 + \int d^3\vec{r}_1 |\psi(\vec{r}_1, \vec{r})|^2$. If the wave functions have no overlap, then the cross terms contribute negligibly, and this gives $P(\vec{r}) \approx |\psi_1(\vec{r})|^2 + |\psi_2(\vec{r})|^2 \approx |\psi_1(\vec{r})|^2$ and the distant electron decouples, despite the fact that the wave function is antisymmetric under exchanging

• Example of N identical spin 1/2 Fermions in a potential well. Suppose that the total spin is $N/2$, i.e. fully symmetric. Then the wave function must be fully antisymmetric, so fill up the first N energy levels (filling the Landau levels). The wave function is a Slater determinant $\psi(\vec{r}_1, \ldots, \vec{r}_N) = N^{-1/2} \det_{ij} \psi_i(\vec{r}_j)$.

• Helium atom:

$$H = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}}.$$

In the approximation where we ignore the last term (electron-electron interaction), $H_0 = H_1 + H_2$, so the energy eigenstates without imposing that the electrons are identical would be products $\psi_{n_1,\ell_1,m_1}(\vec{r}_1)\psi_{n_2,\ell_2,m_2}(\vec{r}_2)$. We need to symmetrize in $\vec{r}_1 \leftrightarrow r_2$ and antisymmetrize in spin, or visa-versa to account for the fact that the electrons are identical.

- Aside on quarks and the strong force (QCD). E.g. baryons with $j = 3/2$ have $|+++\rangle$ for their spin state, which is totally symmetric. If one considers baryons made from u, d, s quarks (the 3 lighter flavors), there is an approximate $SU(3)_F$ symmetry and it is observed that the baryons are in the $(3 \times 3 \times 3)_S$ fully symmetric, i.e. the 10 dimensional representation (recall the 3d SHO degeneracy of the $n = 3$ state). The quarks have $j = 1/2$ each and are Fermions, so the total state must be fully antisymmetric. The strong force $SU(3)_C$ gives the antisymmetry, since $\psi_{c_1}\psi_{c_2}\psi_{c_3}\epsilon^{c_1c_2c_3}$ is a color singlet (color is confined into neutral objects) and fully antisymmetric.