Physics 212b, Ken Intriligator lecture 8, Feb 5, 2018

• Continue with time reversal: $T : t \to -t$, $\vec{x} \to \vec{x}$, $\vec{p} \to -\vec{p}$, $\vec{S} \to -\vec{S}$, $\vec{J} \to -\vec{J}$, $\vec{E} \to \vec{E}$, $\vec{B} \to -\vec{B}$.

This is a symmetry of usual classical physics. Time reversal in QM is an anti-unitary operator. E.g. if $\psi(\vec{x}, t)$ is a solution of the SE, then so is $\psi^*(\vec{x}, -t)$. The complex conjugation is the anti-unitary part. Writing the time reversal operator as T, it must be anti-unitary because we want $T^{-1}U(t)T = U(-t)$, but $T^{-1}HT = H$ if the theory is time-reversal invariant (i.e. $T^{-1}HT \neq -H$). Time reversal acts as e.g. $T^{-1}\vec{x}T = \vec{x}$ and $T^{-1}\vec{p}T = -\vec{p}$, which again forces anti-unitarity, to preserve the commutation relations.

An anti-unitary linear operator acts as $T^{-1}(a\hat{A} + b\hat{B})T = a^*T^{-1}\hat{A}T + b^*T^{-1}\hat{B}T$. In terms of states $T(a|\psi\rangle + b|\chi\rangle) = a^*T|\psi\rangle + b^*T|\chi\rangle$. Can write $T = UK$ where K is complex conjugation. In position space this takes $\vec{x} \to \vec{x}$ and $\vec{p} \to -\vec{p}$ simply because $\vec{x}$ is real and $\vec{p} = -i\hbar\nabla$ is imaginary, so we can take $T = UK$ with $U = 1$ in position space.

$T|\vec{p}\rangle = |-\vec{p}\rangle$ and $T|\vec{x}\rangle = |\vec{x}\rangle$. If $|\psi\rangle = \int d^3\vec{x}|\vec{x}\rangle\langle\vec{x}|\psi\rangle$, then $T|\psi\rangle = \int d^3\vec{x}|\vec{x}\rangle\langle\vec{x}|\psi\rangle^*$.

$T\vec{P} = -\vec{P}T$, since the translation operator commutes with T. Likewise, $T\vec{J} = -\vec{J}T$, since the rotation operator and time reversal commute (and time reversal flips the i in the rotation operator).

$T|\ell, m\rangle = (-1)^m|\ell, -m\rangle$, which is seen in the $Y_{\ell m}$ since T complex conjugates it. More generally, $T^2 = 1$ for integer j.

• The anti-unitary operator T satsfies $\langle T\psi|T\chi\rangle = \langle\psi|\chi\rangle^* = \langle\chi|\psi\rangle$, and likewise $\langle K\psi|K\chi\rangle = \langle\chi|\psi\rangle$.

• Terms in H that respect T: $\vec{p}^2$, $V(\vec{x})$, $\vec{L} \cdot \vec{S}$, $\vec{p} \cdot \vec{S}$. Possible term that are T odd: $p \cdot \vec{x}$, $\vec{S} \cdot \vec{x}$.

• Example, SHO satisfies $TH = HT$, so T is a symmetry. Note that $a = \sqrt{\frac{m\omega}{2\hbar}}(x + ip/m\omega)$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(x - ip/m\omega)$, $[a, a^\dagger] = 1$, and $Ta = aT$ and $Ta^\dagger = a^\dagger T$. So we can take $T|n\rangle = |n\rangle$ for all n. It is expected that the energy eigenstates are T eigenstates, since T is a symmetry and the states are non-degenerate. The fact that they are all T even shows that $\psi_n(x) = \langle x|n\rangle$ are all real.

Theorem: if H is invariant under T and $|n\rangle$ is non-degenerate, then $\langle\vec{x}|n\rangle$ must be real (the phase must be a constant). Follows from fact that $|n\rangle$ and $T|n\rangle$ must be degenerate, and then by assumption must be the same.

For the SHO, we can form states like $(|0\rangle + i|1\rangle)/\sqrt{2}$ that are not energy eigenstates and not T eigenstates, of course.

• Note that $T^2 \neq 1$ for states of odd spin: indeed, $T^2 = (-1)^{2j}$. E.g. consider spin 1/2, so we can represent $\vec{S}$ with $\frac{1}{2}\hbar\sigma$. Writing $T = UK$, note that K flips the sign of S_y but not S_x or S_z. Consider $|\hat{n};+\rangle = e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}|+\rangle$, and then $T|\hat{n};+\rangle = e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}T|+\rangle = \eta e^{-iS_z\alpha/\hbar}e^{-iS_y\beta/\hbar}|-\rangle$ for some phase η. Find $T = \eta e^{-i\pi J_y/\hbar}K$ where K denotes complex conjugation. T^2 is a 2π rotation, so $T^2 = (-1)^{2j}$. Can take $T|j,m\rangle = i^{2m}|j,-m\rangle$.

Let A be an operator that is T even or odd, then $\langle\alpha, j, m|A|\alpha jm\rangle = \pm\langle\alpha, j, -m|A|\alpha.j, -m\rangle$. If $A = T_0^{(k)}$, then we can do a π rotation to convert the $-m$ back to $+m$ on the RHS, getting a factor of $(-1)^k$ from rotating $T_0^{(k)}$ by π. The selection rule implies that time reversal even operators can only have a non-zero expectation value in the $|\alpha jm\rangle$ state if k is even, and likewise for odd operators under T, k must be odd.

• Electricity and magnetism preserve both P and T. Recall $L = L_0 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$ and $\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}$ and $H = (\vec{p} - q\vec{A}/c)^2/2m + q\phi$. Under T, take $\vec{J} \to -\vec{J}$, $\vec{A} \to -\vec{A}$, $\vec{p} \to -\vec{p}$, $\vec{v} \to -\vec{v}$, $\vec{E} \to \vec{E}$, $\vec{B} \to \vec{B}$, preserves e.g. $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$.

• But if $\vec{B}$ is an external magnetic field, that "spontaneously" breaks the symmetries, since then can't have $\vec{B} \to -\vec{B}$. The $\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}$ break T, so $TH \neq HT$. Recall also spin 1/2 in $H = -(ge/2m_ec)\vec{S}\cdot\vec{B}$ and note that it respects both P and T, with both $\vec{S}$ and $\vec{B}$ odd. E.g. $T|+\rangle = |-\rangle$ and the two states have different energy.

Kramers degeneracy: if a system (e.g. a gas or crystal) is made up of atoms that each have an odd number of electrons, then $T^2 = -1$ and there has to be at least a 2-fold degeneracy, i.e. $|n\rangle$ and $T|n\rangle$. Cannot have $T|n\rangle = \eta|n\rangle$, since that would lead to $T^2|n\rangle = +|n\rangle$, incompatible with $T^2 = -1$. The degeneracy can be lifted by placing the system in an external magnetic field. Note T flips the sign of $\vec{B}$, so an external $\vec{B}_{ext}$ breaks T symmetry.