• Last time, symmetry in QM (follow Sakurai chapter 4). Recall time translations via $U(t, t_0) = e^{-i\widehat{H}(t-t_0)/\hbar}$. Space translations are generated by $U(\vec{a}) = e^{-i\vec{a}\cdot\widehat{P}/\hbar}$, which acts as $U(\vec{a})|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$ and rotations are generated by $U(\vec{\phi}) = e^{-i\vec{\phi}\cdot\widehat{J}/\hbar}$. E.g. a rotated version of the operator $\vec{x}$ is $U^\dagger(\vec{\phi})\vec{x}U(\vec{\phi})$ which for infinitesimal ϕ is $\vec{x}' = \vec{x} + \vec{\phi} \times \vec{x}$. Aside on active vs passive transformations.

In general, symmetries are given by *groups*, and continuous symmetries are *Lie groups*, and they can be written as exponentials of *Lie algebras*. Don't worry about these math names if they're unfamiliar. You already know many of the crucial points in the context of a particular example: the rotation group, which is $SU(2)$. The j spin labels what is called the representation, and e.g. addition of angular momentum is an example of decomposing tensor products of representations into irreducible representations. The standard model of particle physics is based on the local gauge symmetry groups $SU(3)_C \times SU(2)_W \times U(1)_Y$. Discrete groups enter in studies of crystals and such. Take Physics 220 if you'd like to learn more about group theory and applications to Nature.

In QM, a continuous symmetry acts infinitesimally as $U(\epsilon) = 1 - i\epsilon G/\hbar + O(\epsilon^2)$. A finite transformation can be regarded as a product of infinitesimally small ones: $U(a) = \lim_{N\to\infty} U(a/N)^N = \lim_{N\to\infty}(1 - iaG/N\hbar)^N = \exp(iaG/\hbar)$. To preserve the inner product norm $\langle\chi|\psi\rangle$, U should be unitary, and hence G should be Hermitian. In classical physics, continuous symmetries lead to conserved quantities via Noether's theorem. In QM, these quantities are the operators G, and U are the corresponding symmetry transformations. They are conserved if G commutes with $U(t) = e^{-iHt/\hbar}$, which is the case if $[G, H] = 0$. Then H and G can be simultaneously diagonalized: eigenstates of G are eigenstates of H and their eigenvalues in the Heisenberg picture do not change in time.

In the language of group theory, the energy eigenstates with eigenvalue E_n must form representations of the symmetry.

For rotational symmetry, the fact that $[J_i, H] = 0$ implies that we can find simultaneous eigenstates $|n; j, m\rangle$ of H, $\vec{J}^2$, and J_z. Under a rotation

$$\mathcal{D}(R)|n; j, m\rangle = \sum_{m'=-j}^{j} |n; j, m'\rangle \mathcal{D}^{(j)}_{m,m'}(R).$$

The states with different m values of course have the same energy, since they can be obtained by acting with $J_\pm = J_x \pm iJ_y$ and $[J_\pm, H] = 0$. Remember e.g. the Wigner-Eckart relation, which is an example of how to use symmetry to obtain relations and selection rules:

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\langle jk; mq|jk; j'm'\rangle\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

The idea is that the product $T_q^{(k)}|\alpha, jm\rangle$ is adding the angular momentum of $T_q^{(k)}$ to that of $|\alpha, jm\rangle$ so the inner product with $\langle \alpha', j'm'|$ is given by a Clebsch Gordon coefficient. The first term on the RHS is a CG coefficient, which is zero unless $m' = q + m$ and $|j - k| \le j' \le j + k$. The last term is independent of m and m'; this is where the symmetry gives some helpful mileage. Example: for a scalar operator S get

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = (2j+1)^{-1/2}\delta_{j'j}\delta_{m'm}\langle \alpha' j'||T^{(k)}||\alpha j\rangle.$$

For a vector operator $\vec{V}$ get $j' - j = 0, \pm 1$ and $m' - m = \pm 1, 0$. Useful in perturbation theory for radiation, as we will see. Another immediate application: a state of spin j cannot have a non-zero expectation value of an operator with angular momentum ℓ unless $\ell \le 2j$. Therefore, a particle of spin zero cannot have a non-zero magnetic dipole moment and a particle with spin $1/2$ cannot have an electric quadrupole moment.

• Recall that the Coulomb potential has an additional degeneracy. Generally we can have $E_{n,\ell}$, but the Coulomb potential has energy E_n for all $\ell = 0, 1 \ldots n - 1$, giving a degeneracy of $\sum_{\ell=0}^{n-1}(2\ell + 1) = n^2$. This larger degeneracy is the consequence of a larger symmetry. In classical physics, this symmetry leads to the conserved Runge-Lenz vector, $\vec{M} = m^{-1}\vec{p} \times \vec{L} - Ze^2 r^{-1}\vec{r}$. The rotation symmetry generated by $\vec{L}$ is $SO(3)$ (or $Spin(3) \cong SU(2)$ if we include spinors), and that obtained by including $\vec{M}$ is $SO(4) \cong SU(2) \times SU(2)$. In QM we modify $\vec{M}$ to make it Hermitian: $\vec{M} = (2m)^{-1}(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}) - (Ze^2/r)\vec{r}$. This satisfies $[\vec{M}, H] = 0$. Since $\vec{M}$ transforms as a vector, it satisfies $[M_i, L_j] = i\hbar\epsilon_{ijk}M_k$. Also, it satisfies $[M_i, M_j] = -i\hbar\epsilon_{ijk}(2/m)HL_k$. Acting on energy eigenstates, we can replace $H \to E$ and define $\vec{N} = (-m/2E)^{1/2}\vec{M}$, with $[N_i, N_j] = i\hbar\epsilon_{ijk}L_k$. Define $\vec{I} = \frac{1}{2}(\vec{L} + \vec{N})$ and $\vec{K} = \frac{1}{2}(\vec{L} - \vec{N})$, and see that they form decoupled $SU(2)$ representations:

$$[I_i, I_j] = i\hbar\epsilon_{ijk}I_k, \qquad [K_i, K_j] = i\hbar\epsilon_{ijk}K_k, \qquad [I_i, K_j] = 0$$

and $[I_i, H] = [K_i, H] = 0$. So energy eigenstates form representations of $SU(2)_I$ and $SU(2)_K$, labeled by i, k which can be integer or half-integer, and the degeneracy is $(2i + 1)(2k + 1)$. Also there is another relation: $\vec{I}^2 - \vec{K}^2 = \vec{L} \cdot \vec{N} = 0$, which requires $i = k$. Also

$$\vec{I}^2 + \vec{K}^2 = \tfrac{1}{2}(\vec{L}^2 + \vec{N}^2) = \tfrac{1}{2}(\vec{L}^2 - \frac{m}{2E}\vec{M}^2),$$

with $\vec{M}^2 = \frac{2}{m}H(\vec{L}^2 + \hbar^2) + Z^2e^4$. These imply $2k(k + 1)\hbar^2 = \frac{1}{2}(-\hbar^2 - \frac{m}{2E}Z^2e^4)$ and thus $E = -(mZ^2e^4/2\hbar^2)n^{-2}$ with $n = 2k + 1$. The degeneracy of $(2k + 1)^2$ is thus a degeneracy of n^2. The symmetry completely determines the hydrogen atom energy levels and degeneracy, with no need to solve the Schrodinger equation.