Physics 212b, Ken Intriligator lecture 3, Jan 17, 2018

• Gauge invariance in QM: consider a charged particle in electric and magnetic fields. Recall from classical mechanics that $L = L_0 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$, so $\vec{p} = \vec{p}_0 + \frac{q}{c}\vec{A}$. Note gauge invariance. Aside on relativity: $\int dt(-c\phi + \vec{v}\cdot\vec{A})(q/c) = -q/c\int A_\mu dx^\mu$ is Lorentz invariant. Then $H = (\vec{p} - q\vec{A}/c)^2/2m + V(\vec{x}) + q\phi$. Replace $\vec{p} \to -i\hbar\nabla$ in QM in position space. Gauge invariance: $\vec{A} \to \vec{A} + \nabla f$, $\phi \to \phi - \partial f/c\partial t$ preserves $\vec{E} = -\nabla\phi - \partial\vec{A}/c\partial t$. Gauge invariance is sometimes thought of as a symmetry, but it is best thought of as coming from using redundant variables: it's not just that different gauge choices lead to equivalent physics, but that there is no physical difference between them. In classical physics, gauge invariance is seen in that physical effects only depend on $\vec{E}$ and $\vec{B}$. In QM, it is more subtle: gauge transformations affect the phase of the wavefunction $\psi \to e^{iqf/\hbar c}\psi$, but is an exact symmetry of any and all physics. Fundamental in high energy physics: forces = gauge symmetries. In the path integral, we integrate over gauge field configurations modulo gauge transformations.

• QM in electric and magnetic fields: $\vec{p} = \partial_{\vec{v}}L = m\vec{v} + \frac{q}{c}\vec{A}$. If there is translation symmetry, this is the $\vec{p}$ that is conserved (actually, we should include the field momentum $\sim \int d^3\vec{x}\vec{E}\times\vec{B}$, and the $\vec{A}$ term in $\vec{p}$ corresponds to the fact that charged particles can transfer momentum to the fields). When we quantize, this is the $\vec{p}$ that becomes an operator, satisfying $[x_n, p_m] = i\hbar\delta_{nm}$. When we go to position space, we replace this $\vec{p}$, not $m\vec{v}$ with $-i\hbar\nabla$. The Hamiltonian is $H = (\vec{p} - q\vec{A}/c)^2/2m + V(\vec{x}) + q\phi$, and the fact that $\vec{A}$ drops out if expressed in terms of $\vec{v}$ (which should not be done) is the statement that magnetic fields do not do work.

• Computing the kernel $K(\vec{x}_2, t_2; \vec{x}_1, t_1)$ for a small time step again reproduces the statement of the Feynman path integral: the kernel is $\sim e^{iS/\hbar} \sim \exp(\frac{iq}{\hbar c}\int_{\text{path}} d\vec{x}\cdot\vec{A})$.

• Aharanov-Bohm / Dirac effect: use $\psi \sim e^{iS/\hbar}$ and compare interference on two paths, on two sides of solenoid: $\psi_1/\psi_2 = e^{i(S_1 - S_2)/\hbar}$ and note that $(S_1 - S_2) = \oint (q/c)\vec{A}\cdot d\vec{\ell} = q\Phi/c$, where Φ is the magnetic flux. So e.g. $\psi_1 = \psi_2$ if $q\Phi = 2\pi\hbar c n$, which if we set $\Phi = 4\pi q_{mag}$ is Dirac's quantization rule. Magnetic monopoles could be the explanation of electric charge quantization.

• Symmetry in QM (follow Sakurai chapter 4). Recall time translations via $U(t, t_0) = e^{-i\widehat{H}(t-t_0)/\hbar}$. Space translations are generated by $U(\vec{a}) = e^{-i\vec{a}\cdot\widehat{\vec{P}}/\hbar}$, which acts as $U(\vec{a})|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$ and rotations are generated by $U(\vec{\phi}) = e^{-i\vec{\phi}\cdot\widehat{\vec{J}}/\hbar}$. E.g. a rotated version of the operator $\vec{x}$ is $U^\dagger(\vec{\phi})\vec{x}U(\vec{\phi})$ which for infinitesimal ϕ is $\vec{x}' = \vec{x} + \vec{\phi}\times\vec{x}$. Aside on active vs passive transformations.

In general, symmetries are given by *groups*, and continuous symmetries are *Lie groups*, and they can be written as exponentials of *Lie algebras*. Don't worry about these math names if they're unfamiliar. You already know many of the crucial points in the context of a particular example: the rotation group, which is $SU(2)$. The j spin labels what is called the representation, and e.g. addition of angular momentum is an example of decomposing tensor products of representations into irreducible representations. The standard model of particle physics is based on the local gauge symmetry groups $SU(3)_C \times SU(2)_W \times U(1)_Y$. Discrete groups enter in studies of crystals and such. Take Physics 220 if you'd like to learn more about group theory and applications to Nature.

In QM, a continuous symmetry acts infinitesimally as $U(\epsilon) = 1 - i\epsilon G/\hbar + O(\epsilon^2)$. A finite transformation can be regarded as a product of infinitesimally small ones: $U(a) = \lim_{N\to\infty} U(a/N)^N = \lim_{N\to\infty}(1 - iaG/N\hbar)^N = \exp(iaG/\hbar)$. To preserve the inner product norm $\langle\chi|\psi\rangle$, U should be unitary, and hence G should be Hermitian. In classical physics, continuous symmetries lead to conserved quantities via Noether's theorem. In QM, these quantities are the operators G, and U are the corresponding symmetry transformations. They are conserved if G commutes with $U(t) = e^{-iHt/\hbar}$, which is the case if $[G, H] = 0$. Then H and G can be simultaneously diagonalized: eigenstates of G are eigenstates of H and their eigenvalues in the Heisenberg picture do not change in time.

In the language of group theory, the energy eigenstates with eigenvalue E_n must form representations of the symmetry.

For rotational symmetry, the fact that $[J_i, H] = 0$ implies that we can find simultaneous eigenstates $|n; j, m\rangle$ of H, $\vec{J}^2$, and J_z. Under a rotation

$$\mathcal{D}(R)|n; j, m\rangle = \sum_{m'=-j}^{j} |n; j, m'\rangle \mathcal{D}^{(j)}_{m,m'}(R).$$

The states with different m values of course have the same energy, since they can be obtained by acting with $J_\pm = J_x \pm iJ_y$ and $[J_\pm, H] = 0$. Remember e.g. the Wigner-Eckart relation, which is an example of how to use symmetry to obtain relations and selection rules:

$$\langle\alpha', j'm'|T^{(k)}_q|\alpha, jm\rangle = (2j+1)^{-1/2}\langle jk; mq|jk; j'm'\rangle\langle\alpha' j'||T^{(k)}||\alpha j\rangle.$$