Physics 212b, Ken Intriligator lecture 2, Jan 10, 2018

• Lightening review of WKB (Wentzel, Kramers, Brillouin) (will cover it in more detail if it was not already seen last quarter). For high momentum, $\psi_E(x)$'s wiggles are smaller than $V(x)$'s wiggles, so can approximate solutions via $V(x) \approx$ constant and then add successive corrections. Write the time-indep SE in terms of $k(x) = \sqrt{2m(E-V(x))/\hbar^2}$ or $k(x) \equiv -i\sqrt{2m(V(x)-E)/\hbar^2}$ in $E < V$ and $E > V$ regions respectively, so

$$\psi_E'' + k(x)^2\psi_E(x) = 0.$$

Take $\psi_E(x) \equiv e^{iW(x)/\hbar}$ to get $\frac{2}{3}|z|^{3/2} = \hbar^{-1}\int^x dx'\sqrt{2m(E-V(x')}$.

$$i\hbar W'' - (W')^2 + \hbar^2 k^2 = 0.$$

So for $\hbar|W''|^2 \ll |W'|^2$ we end up with $W_0'(x) = \pm\hbar k(x)$. Define $W(x) = \sum_{n=0}^{\infty}\hbar^n W_n(x)$ and plug back in to get an iterative equation for W_{n+1} in terms of W_n. In particular, $W_0' + W_1' = \pm\sqrt{\hbar^2 k(x)^2 + i\hbar W_0''}$ where expanding the square-root and integrating gives

$$\psi_E \approx e^{i(W_0+\hbar W_1)/\hbar)} \approx |k(x)|^{-1/2}\exp[\pm i\int^x dx' k(x')].$$

Note that $|\psi_E|^2 \approx |k(x)|^{-1} \sim 1/v(x)$, which agrees with what one might call the classical likelihood of finding a particle with velocity v in some region dx, since $dx/v = dt$ is the time that it spends in that region.

• We have to patch together these solutions across the values of x where $E = V$; in those vicinities can approximate in terms of the linear potential, with the Airy function. Suppose that there are classical turning points at $x = x_1$ and $x = x_2$, so the classical motion is for $x_1 \leq x \leq x_2$, which is called region II. Regions I and III are the classically forbidden regions $x < x_1$ and $x > x_2$. Match the WKB solution in region II to the asymptotic behavior of the Airy function at the turning point, where V is approximately linear: $Ai(z) \to z^{-1/4}(2\sqrt{\pi})^{-1}e^{-2z^{3/2}/3}$ for $z \to \infty$ and $Ai(z) \to |z|^{-1/4}\pi^{-1/2}\cos(2/3|z|^{3/2} - \pi/4)$ for $z \to -\infty$. Match the $z \to \infty$ behavior to $\psi_{I,III}$ to get

$$\psi_{E,I\to II} \to 2(E-V(x))^{-1/4}\cos\left(\hbar^{-1}\int_{x_1}^{x} dx'\sqrt{2m(E-V(x'))} - \pi/4\right),$$

$$\psi_{E,III\to II} \to 2(E-V(x))^{-1/4}\cos\left(-\hbar^{-1}\int_{x}^{x_2} dx'\sqrt{2m(E-V(x'))} + \pi/4\right),$$

and the two must agree. So the argument of the cos must differ by $n\pi$. The upshot is that, if x_1 and x_2 are two classical turning points, these approximations lead to $\int_{x_1}^{x_2} dx\sqrt{2m[E-V(x)]} = (n+\frac{1}{2})\pi\hbar$, like the Born Sommerfield Wilson quantization $\oint pdq = 2\pi n\hbar$. Note that for e.g. the SHO the classical solution is $x = A\cos(\omega t+\phi)$, $p = m\dot{x} = -m\omega A\sin(\omega t+\phi)$, $\oint pdq = \int_0^{2\pi/\omega} A^2 m\omega^2 \sin^2(\omega t+\phi)dt = \pi m\omega A^2 = 2\pi E/\omega$, so the WKB quantization rule gives $E_n = (n+\frac{1}{2})\hbar\omega$, so in this case it gives the exact result. More generally, it gives a good approximation for E_n when $n \gg 1$.

• Also, tunneling through a barrier: probability $\sim e^{-2\int_{x_1}^{x_2}\sqrt{2m(V_{eff}(x)-E)}dx/\hbar}$, where x here could also denote the radial direction of a 3d system.

• Now connect to the path integral, using

$$\psi(x,t) = \int dx' K(x,t;x',t')\psi(x',t'),$$

and the saddle point approximation of the path integral gives

$$K(x,t;x',t') \approx Ae^{iS_{cl}/\hbar} = Ae^{-iE(t-t')/\hbar}\exp(\frac{i}{\hbar}\int_{t'}^{t} dt 2T) = Ae^{-iE(t-t')/\hbar}\exp(\frac{i}{\hbar}\int_{x'}^{x} p(x)dx).$$

where we used $L = T - V = 2T - E$, and we can take E out of the integral since it is conserved. The $|p|^{-1/2}$ prefactor in the WKB wavefunction comes from doing the Gaussian integral for quadratic functions in the Tayler expansion of S around the saddle point extremum, i.e. around the classical path. So we find that the approximate K is consistent with $\psi_E \approx e^{\frac{i}{\hbar}\int^x pdx}$: the x' dependence cancels (the $\int dx'$ is damped by the exponential falloff of $\psi_E(x)$ so really $\int dx' \to const$, that is absorbed into A.)