Physics 212b, Ken Intriligator lecture 16, Mar 12, 2018

• Last time: an example of an $H(t)$ that can be solved exactly (Rabi). Two state system with $H = H_0 + H_t(t)$ and $H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$ and $H_1 = \gamma e^{i\omega t}|1\rangle\langle 2| + h.c.$. Suppose $|\psi(t_0 = 0)\rangle = |1\rangle$. Find the probability of finding the system in state $|2\rangle$ at a later time t as given by Rabi's formula:

$$|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega - \omega_{21})^2/4} \sin^2 \Omega t,$$

where $\omega_{21} \equiv (E_2 - E_1)/\hbar$ and $\Omega \equiv \sqrt{(\gamma^2/\hbar^2) + \frac{1}{4}(\omega - \omega_{21})^2}$. Resonant enhancement of the transition amplitude when $\omega \approx \omega_{21}$. Of course $|c_1(t)|^2 = 1 - |c_2(t)|^2$. The times of growing $|c_2(t)|^2$ are absorption and those of decreasing $|c_2(t)|^2$ are emission.

Example spin magnetic resonance: $\vec{\mu} = e\vec{S}/m_e c$ in $\vec{B} = B_0\hat{z} + B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$. Identify $\omega_{21} = |e|B_0/m_e c$ and $\gamma = |e|\hbar B_1/2m_e c$.

• Maser (Microwave amplification of stimulated emission of radiation): $\vec{E} = E_0\hat{z}\cos\omega t$, e.g. for the ammonia molecule NH_3. Two parity states $|\pm\rangle$ (or R and L) such that $\omega = (E_- - E_+)/\hbar$ is in the microwave region, $H_1 = -\vec{\mu}_{el}\cdot\vec{E}$.

• Time dependent perturbation theory. Some examples to keep in mind: (i) turning on a perturbation for $t > t_0$: $H(t) = H_0 + H_1\Theta(t - t_0)$; (ii) turning on a perturbation for $t_0 < t < t_1$: $H(t) = H_0 + (\Theta(t - t_0) - \Theta(t - t_1))$. Another case is a harmonic perturbation, e.g. $H_1(t) = H_1\cos\omega t$, which can lead to stimulated emission or absorption. Some other changes that are not necessarily small perturbations are sudden or adiabatic changes (e.g. particle in a box that suddenly or adiabatically expands).

• The time dependence can be written in terms of the interaction or Dirac picture. Recall from lecture 1: $i\hbar\partial_t U(t, t_0) = HU(t, t_0)$. For the case of time independent H, this integrates to $U(t, t_0) = \exp(iH_0(t - t_0)/\hbar)$. In the Schrodinger picture one takes $|\psi_S(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, which then solves the Schrodinger equation. In the Heisenberg picture one instead puts the time evolution in the operators: $\mathcal{O}_H(t) = U^\dagger(t, t_0)\mathcal{O}U(t, t_0)$, which then evolve according to the quantum version of the Hamilton equations: $\frac{d}{dt}\mathcal{O}_H = (i\hbar)^{-1}[\mathcal{O}_H, H] + \partial_t\mathcal{O}_H$. Either way, $P_{i\to f} = |\langle f|U(t_f, t_i)|i\rangle|^2$.

For time-dependent perturbation theory it is convenient to use the interaction or Dirac picture, where H_0 gives the time evolution of the operators and H_1 gives time evolution of the states. Write time-dependent Schrodinger equation in terms of $|\psi_I(t)\rangle = e^{iH_0 t/\hbar}|\psi(t)\rangle$ and $H_1(t) = e^{iH_0 t/\hbar}H_1 e^{-iH_0 t/\hbar}$. This is going to the Heisenberg picture for the H_0 part only, and leads to $i\hbar\partial_t|\psi_I(t)\rangle = H_1(t)|\psi_I(t)\rangle$, i.e. the H_1 part is gives a Schrodinger picture time evolution. This hybrid is called the interaction representation. Integrate to obtain $|\psi_I(t)\rangle = |\phi_I(t_0)\rangle + (i\hbar)^{-1}\int_{t_0}^{t} dt' H_1(t')|\psi_I(t')\rangle$. To first order in H_1 replace $|\psi_I(t)\rangle$ with $|\psi_I(t_0)\rangle$ in the last term, and then iterate that to obtain

$$|\psi_I(t)\rangle = T(\exp(-\frac{i}{\hbar}\int_{t_0}^{t} H_1(t')dt'))|\psi_I(t_0)\rangle.$$

Here T is time-ordering.

- Write the SE: $i\hbar\partial_t|\psi(t)\rangle = (H_0 + H_1(t))|\psi(t)\rangle$ and expand in the basis of eigenstates of H_0: $|\psi(t)\rangle = \sum_n d_n(t)e^{-iE_n^{(0)}t}|n^0\rangle$, and then the SE gives $0 = \sum_n(i\hbar\dot{d}_n(t) - H^1(t)d_n(t))e^{-iE_n^0 t/\hbar}|n^0\rangle$ and then get $i\hbar\dot{d}_f = \sum_n\langle f^0|H_1(t)|n^0\rangle e^{i\omega_{fn}t}d_n(t)$, where $\omega_{fn} = (E_f^0 - E_n^0)/\hbar$. If at time $t = 0$ the system is in state $|i^0\rangle$ then the first order result is

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle f^0|H_1(t')|i^0\rangle e^{i\omega_{fi}t'}dt' + \mathcal{O}(H_1^2).$$