1/21/18 Physics 212b Homework 2, due Monday Jan 29

1. Consider a particle of mass m that is on a circular hoop of radius R. Let $\phi \sim \phi + 2\pi$ be the angular generalized coordinate for the particle. The particle has charge q and there is a constant, uniform magnetic field perpendicular to the plane of the hoop. The particle is otherwise free.

   (a) Write the Lagrangian $L(\phi, \dot{\phi})$ and find the conjugate momentum p_ϕ and Hamiltonian $H(\phi, p_\phi)$. To include the effect of $\vec{B}$ you'll need to find $A_{\widehat{\phi}}$ and please express your answers in terms of $\Theta \equiv q\Phi_B = qB\pi R^2$.

   (b) The quantum state $\phi(\phi) \equiv \langle \phi | \psi \rangle$ should be periodic in ϕ. Find the energy eigenstates and eigenvalues.

   (c) Verify that the physics of the above answer is periodic under $\Theta \to \Theta + 2\pi$.

2. To get a flavor for the Feynman path integral, this exercise will be to consider K_{SHO} (see the online lecture notes for lecture 1).

   (a) Write the expression for the action of the simple harmonic oscillator as a sum, instead of the usual integral, replacing dt with Δt.

   (b) Write Feynman's prescription for $K = \langle x_f, t_f | x_o, t_o \rangle$ as an infinite product of integrals, and notice that they are all coupled Gaussian integrals.

   (c) Now suppose that $t_f - t_o$ is very small, corresponding to two Δt steps. In this case, there is only one Gaussian integral to do, over the position x_i of the particle at the intermediate time t_i. Do that Gaussian integral and verify that the answer agrees with that given in the online lecture notes when $t_f - t_o$ (called $t_2 - t_2$ in the notes) is infinitesimal.

3. Consider a particle of mass m in a 1d potential $V(x) = -\alpha\delta(x)$, with α a positive constant (so it's a potential well, not barrier). Find the bound state energy $E < 0$ and its energy eigenfunction. Note the parity of the eigenfunction.

4. Consider a particle of mass m in a 1d potential that is that of a SHO for $x > 0$, and an infinite barrier for $x < 0$. Find the ground state energy and $\langle x^2 \rangle$ in the ground state.

5. Consider a particle of mass m in a 1d potential $V(x) = -V_0\Theta(a - |x|)$, where $\Theta(x)$ is the step function ($\Theta(x > 0) = 1$, and $\Theta(x < 0) = 0$, and V_0 positive (so it's a potential well). Suppose that V_0 is infinitesimal, so you can linearize in V_0. Show that there is an even parity bound state, and determine its energy. Is there an odd parity bound state?

6. Consider a particle of mass m with the 1d potential barrier $V(x) = V_0\Theta(a - |x|)$, and suppose that the energy is below the barrier height, $0 < E < V_0$. Take the wave function for $x < a$ to be $\psi_L(x) = e^{i(px-Et)/\hbar} + Re^{-i(px+Et)/\hbar}$ with $p > 0$, and that for $x > a$ to be $\psi_R(x) = Te^{i(px-Et)/\hbar}$. R and T are called the reflection and transmission coefficients.

(a) Write down the wave function within the barrier region, and the conditions that it must satisfy.

(b) Even though the region $|x| < a$ is classically excluded, we can compute a classical action as follows. Do a change of variables $t = \pm i\tau$ in the action and the Lagrangian. I will leave it to you to check which sign is correct, such that $e^{iS/\hbar}$ becomes an exponentially damped factor. Show that the Euler-Lagrange equations for $x(\tau)$ have a solution in the classically forbidden region. Write the solution such that it connects $x(\tau = 0) = -a$ and $x(\tau = \tau_0) = +a$. Compute the action for this solution (the answer is imaginary). Now relate τ_0 to E and the potential by thinking about the path in imaginary time.

(c) Compute and compare T and $e^{iS_{cl}}$ where S_{cl} is the imaginary action of part (b).

7. Let $\mathcal{D}(R)$ be the rotation operator.

(a) What is the time reversed state corresponding to $\mathcal{D}(R)|j, m\rangle$? (b) Using the properties of time reversal and rotation, prove that

$$\mathcal{D}^{(j)}_{m'm}(R) = (-1)^{m-m'}\mathcal{D}^{(j)}_{-m',-m}(R).$$

(c) Prove that time reversal acts as $T|j, m\rangle = (i)^{2m}|j, -m\rangle$.