212b Homework 1, due 1/22/18

1. (Based on Sakurai 2.33) Derive an explicit expression for the momentum space propagator $\tilde{K}(\vec{p}_2, t_2; \vec{p}_1, t_1) \equiv \langle \vec{p}_2 | U(t_2, t_1) | \vec{p}_1 \rangle$ for the case of a free particle. You can use the usual description of QM. As usual, we define $\tilde{K} = 0$ if $t_2 < t_1$. Derive it directly in momentum space, and also verify that it is related to the $\tilde{K}(\vec{x}_2, t_2; \vec{x}_1, t_1)$ given in class by Fourier transforming.

2. Using your answer from the previous question, Fourier transform $t_{1,2} \to E_{1,2}$ to show that a free nonrelativistic particle has propagator

$$\tilde{K}_{free}(\vec{p}_2, E_2; \vec{p}_1, E_1) = (2\pi\hbar)^4 \delta^3(\vec{p}_2 - \vec{p}_1)\delta(E_2 - E_1)\frac{i}{E_1 - \frac{p_1^2}{2m} + i\epsilon}.$$

The ϵ here comes from something similar to the integral $\int_0^\infty e^{i\omega\tau} d\tau = \lim_{\epsilon\to 0^+} i(\omega + i\epsilon)^{-1}$, where ω is given a small, positive imaginary part to make the integral converge.

3. Take space to be a 1d periodic box of length $2\pi R$, with $x \sim x + L$.

(a) Note that the momentum of a free particle of mass m on this space is quantized, $p = p_n = ...$ (write it down).

(b) Derive an expression for $K(x_2, t_2; x_1, t_1)$ for a free particle of mass m on this space in terms of the Jacobi theta function $\theta(z; \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi(n^2\tau + 2nz))$.

4. (Sakurai 2.38) Consider the Hamiltonian of a spinless particle of charge e. In the presence of a static magnetic field, the interaction terms can be generated by $\vec{p}_{op} \to \vec{\Pi} = \vec{p}_{op} - e\vec{A}/c$. Suppose $\vec{B} = B\hat{z}$ for constant B and show that this leads to the correct expression between $\vec{\mu}_{orbital} = (e/2mc)\vec{L}$ and $\vec{B}$. Show that there is an extra term proportional to $B^2(x^2 + y^2)$ and comment on its physical significance.

5. a) Verify (using the Schrodinger equation) that the probability current is still conserved for a charged particle in a magnetic field if we modify $\vec{j}$ to $\vec{j} = -i(\hbar/2m)\psi^* \nabla\psi - \psi \nabla\psi^*) - (q/mc)\psi^*\psi\vec{A}$.

(b) Verify that $\vec{j}$ is invariant under a gauge transformation $\vec{A} \to \vec{A} + \nabla f$, $\psi(\vec{x}, t) \to e^{\pm iqf/\hbar c}\psi(\vec{x}, t)$; check which sign works, and verify also that doing this transformation in both the Schrodinger equation and $\psi(\vec{x}, t)$, that f drops out.

6. (Sakurai 2.39) Evaluate $[\Pi_x, \Pi_y]$ for $\vec{B} = B\hat{z}$, with B constant, compare to the 1d SHO, and show it leads immediately to energy levels $E_{k,n} = \frac{p_z^2}{2m} + |eB\hbar/mc|(n + \frac{1}{2})$.