Physics 225b, Homework 1 solutions

1. (Aside: consider the geodesics.) The geodesic equation is

$$\frac{d^2x^A}{d\lambda^2}+\Gamma^A_{BC}\frac{dx^B}{d\lambda}\frac{dx^C}{d\lambda}=0.$$

The non-zero Chirstoffel connection components are

$$\Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta,\quad \Gamma^\phi_{\phi\theta}=\Gamma^\phi_{\theta\phi}=\cot\theta.$$

So we can write the geodesic equation as

$$\frac{d^2\theta}{d\lambda^2}-\sin\theta\cos\theta\frac{d\phi}{d\lambda}\frac{d\phi}{d\lambda}=0,\qquad \frac{d^2\phi}{d\lambda^2}+2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda}=0.$$

So we see that $\phi=$ constant solves the 2nd eqn, and the first is then solved too provided that $d^2\theta/d\lambda^2=0$. If we instead set $\theta=$ constant, the first can only be solved if $\sin\theta=0$ or $\cos\theta=0$, since otherwise $d\phi/d\lambda=0$ and there's no motion whatsoever. The choice $\sin\theta=0$ is bad too, since then there's no non-trivial ϕ motion, so the only solution is $\theta=\pi/2$.

We want to solve (not necessarily on a geodesic)

$$\frac{dV^\theta}{d\lambda}-\sin\theta\cos\theta V^\phi\frac{d\phi}{d\lambda}=0,\quad \frac{dV^\phi}{d\lambda}+\cot\theta V^\theta\frac{d\phi}{d\lambda}+\cot\theta V^\phi\frac{d\theta}{d\lambda}=0.$$

Which for $\theta=$ constant we can write as

$$\frac{dV^\theta}{d\phi}=\sin\theta\cos\theta V^\phi,\qquad \frac{dV^\phi}{d\phi}=-\cot\theta V^\theta.$$

The solution for constant θ, satisfying the condition $V^A(\phi=0)=\frac{d}{d\theta}$ is

$$V^\theta=\cos(\cos\theta\phi)=\cos(2\pi\cos\theta),\quad V^\phi=-\csc\theta\sin(\cos\theta\phi)=-\csc\theta\sin(2\pi\cos\theta),$$

where we evaluated it for $\phi=2\pi$.

2. $\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\rho}V^\rho$, and the non-zero Christoffel coefficients are (with $a\equiv t^{2/3}$)

$$\Gamma^0_{ij}=a\dot a\delta_{ij}=\frac{2}{3}t^{1/3}\delta_{ij},\quad \Gamma^i_{j0}=\Gamma^i_{0j}=\frac{\dot a}{a}\delta^i_j=\frac{2}{3t}\delta^i_j.$$

So the non-zero elements of $\nabla_\mu V^\nu$ are

$$\nabla_0V^0=10t,\ \nabla_0V^1=21t^2+\frac{14}{3}t^2,\ \nabla_1V^0=\frac{14}{3}t^{10/3},\ \nabla_1V^1=\nabla_2V^2=\nabla_3V^3=\frac{10t}{3}.$$

3. Can get the Christoffel connection from the geodesic equation, obtained via stationary proper time. Varying t gives

$$\frac{d}{d\lambda}\left((1+Cx)^2\frac{dt}{d\lambda}\right) = (1+Cx)^2\frac{d^2t}{d\lambda^2} + 2C(1+Cx)\frac{dx}{d\lambda}\frac{dt}{d\lambda} = 0.$$

Varying x gives

$$\frac{d^2x}{d\lambda}^2 + C(1+Cx)\frac{dt}{d\lambda}\frac{dt}{d\lambda} = 0.$$

Varying y gives $d^2y/d\lambda^2 = 0$ and $d^2z/d\lambda^2 = 0$. We thus obtain:

$$\Gamma^0_{01} = \Gamma^0_{10} = C(1+Cx)^{-1}, \quad \Gamma^1_{00} = C(1+Cx),$$

with all others zero. Staring at

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu),$$

potentially non-zero components are e.g.:

$$R^0{}_{110} = \partial_1\Gamma^0_{10} + \Gamma^0_{10}\Gamma^0_{01} = -C^2(1+Cx)^{-2} + C^2(1+Cx)(1+Cx) = 0.$$

Likewise, all other possibly non-zero terms turn out to actually vanish! Since $R^\rho{}_{\sigma\mu\nu} = 0$, this space is actually flat, and hence nearby geodesics do not deviate from each other. The fact that it's flat can be exhibited by finding coordinates $x^{\mu'}$ such that $g_{\mu'\nu'} = \eta_{\mu'\nu'}$ (as usual, $\eta = diag(-1,1,1,1)$). You can check that $t' = (C^{-1}+x)\sinh(Ct)$, $x' = (C^{-1}+x)\cosh(Ct)$, $y' = y$, $z' = z$ does the trick: $ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2$. Note that the point $x = 0$ maps to $t' = C^{-1}\sinh(Ct)$ and $x' = C^{-1}\cosh(Ct)$, which is the worldline of a particle with uniform acceleration C along the x' axis. The given metric is simply flat spacetime, as seen by a uniformly accelerating observer. This is sometimes called "Rindler space."

4. The non-zero Chirstoffel symbols are $\Gamma^0_{ij} = a\dot{a}\delta_{ij}$, $\Gamma^i_{j0} = a^{-1}\dot{a}\delta^i_j$. This gives e.g.

$$R^0{}_{i\mu\nu} = \delta_{\mu 0}\partial_t\Gamma^0_{i\nu} + \Gamma^0_{\mu\lambda}\Gamma^\lambda_{\nu i} - (\mu \leftrightarrow \nu) = \delta_{\mu 0}\partial_t(a\dot{a})\delta_{i\nu} + \delta_{\nu 0}\delta^\lambda_i\delta_{\lambda\mu}a\dot{a}a^{-1}\dot{a} - (\mu \leftrightarrow \nu).$$

So $R^0{}_{ijk} = 0$ and

$$R^0{}_{i0j} = \partial_t(a\dot{a})\delta_{ij} - \Gamma^0_{j\lambda}\Gamma^\lambda_{0i} = (\dot{a}^2 + a\ddot{a})\delta_{ij} - a\dot{a}\delta_{\lambda j}a^{-1}\dot{a}\delta^\lambda_i = a\ddot{a}\delta_{ij}.$$

We also have

$$R^i{}_{j\mu\nu} = \Gamma^i_{\mu\lambda}\Gamma^\lambda_{\nu j} - (\mu \leftrightarrow \nu) = \Gamma^i_{\mu 0}\Gamma^0_{\nu j} - (\mu \leftrightarrow \nu) = \delta^i_\mu \delta_{j\nu}\dot{a}^2 - (\mu \leftrightarrow \nu)$$

These are the only independent non-zero component; all other non-zero components are related to them by the various symmetries of the Riemann tensor.

We then compute $R_{i0} = 0$ and

$$R_{00} = R^i{}_{0i0} = g^{ij}R_{j0i0} = a^{-2}R_{0i0i} = -a^{-2}R^0{}_{i0i} == -3\ddot{a}/a.$$

$$R_{ij} = R^0{}_{i0j} + R^k{}_{ikj} = a\ddot{a}\delta_{ij} + (\delta^k_k\delta_{ij} - \delta^k_j\delta_{ik})\dot{a}^2 = (a\ddot{a} + 2\dot{a}^2)\delta_{ij}$$

and

$$R = -R_{00} + a^{-2}R_{ij}\delta^{ij} = 3\frac{\ddot{a}}{a} + 3a^{-2}(a\ddot{a} + 2\dot{a}^2) = 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}.$$

5. The action is $S = S_{EH} + S_{M\beta}$, where $S_{EH}\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$ is the usual Einstein Hllbert action, and

$$S_{M\beta} = \int d^4x\sqrt{-g}\left(F_{\kappa\lambda}F_{\rho\sigma}(-\frac{1}{4}g^{\kappa\rho}g^{\lambda\sigma} + \beta R^{\kappa\rho}g^{\lambda\sigma}) + A_\kappa J_\lambda g^{\kappa\lambda}\right)$$

is the Maxwell action, with the extra β term.

The gravity EOM (eqns of motion) come from $\delta S/\delta g^{\mu\nu} = 0$, and the electricity and magnetism EOM come from $\delta S/\delta A^\mu = 0$. Varying the metric gives terms discussed in class for S_{EH}, to which we should add

$$\begin{aligned} -2\frac{1}{\sqrt{-g}}\frac{\delta S_{M\beta}}{\delta g^{\mu\nu}} = &\left(F_{\kappa\lambda}F_{\rho\sigma}(-\frac{1}{4}g^{\kappa\rho}g^{\lambda\sigma} + \beta R^{\kappa\rho}g^{\lambda\sigma}) + A_\kappa J_\lambda g^{\kappa\lambda}\right)g_{\mu\nu} \\ &+ F_{\mu\lambda}F_{\nu\sigma}(g^{\lambda\sigma} - 2\beta R^{\lambda\sigma}) - 2A_{(\mu}J_{\nu)} \\ &- 2\beta F_{\kappa\lambda}F_{\rho\sigma}g^{\lambda\sigma}(\nabla_\eta(\frac{\delta\Gamma^\eta_{\kappa\rho}}{\delta g^{\mu\nu}}) - \nabla_\kappa(\frac{\delta\Gamma^\eta_{\eta\rho}}{\delta g^{\mu\nu}})), \end{aligned}$$

where the first line comes from $\delta\sqrt{-g}$, the second from $\delta g^{\mu\nu}$ and the third from $\delta R^{\kappa\rho}$.

(a) Let's first just compute the stress tensor for $\beta = 0$, which is give by the above variation with β set to zero:

$$T^{EM}_{\mu\nu} = F_{\mu\lambda}F_{\nu\sigma}g^{\lambda\sigma} - \frac{1}{4}g_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda} - 2A_{(\mu}J_{\nu)} + g_{\mu\nu}A_\lambda J^\lambda.$$

For $J^\mu = 0$

For part (b), we can see how Maxwell's eqn is altered by writing the Euler-Lagrange eqns. for δA_ν variations:

$$\delta S = \int d^4x\sqrt{-g}\left[(-F^{\mu\nu} + 4\beta R^{\kappa\mu}F_{\kappa\lambda}g^{\lambda\nu})\partial_\mu \delta A_\nu + J^\mu \delta A_\mu\right].$$

Integrating by parts, we get

$$\frac{1}{\sqrt{-g}}\partial_\mu[\sqrt{-g}((F^{\mu\nu} + 4\beta R^{\kappa[\mu}F_{\lambda\kappa}g^{\nu]\lambda})] = J^\nu.$$

The current must still be conserved, since the modified action is still gauge invariant under $A_\mu \to A_\mu + \partial_\mu f$. This follows from the above modified Maxwell eqns,

$$\nabla_\nu J^\nu = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}J^\nu) = \frac{1}{\sqrt{-g}}\partial_\mu\partial_\nu[\sqrt{-g}((F^{\mu\nu} + 4\beta R^{\kappa[\mu}F_{\lambda\kappa}g^{\nu]\lambda})] = 0,$$

where the first equality (as discussed in lecture) is a general property of 4-divergences which can be shown using the expressions for the Christoffel connection, the second = uses the above modified Maxwell eqn, and the third = uses the fact that the expression in the $[\cdots]$ is antisymmetric in $\mu \leftrightarrow \nu$. The above modified Maxwell action violates the equivalence principle assumption of Einstein, since it would allow one to notice gravity effects (vie measuring electric and magnetic fields) even in a free-falling frame, since even in a local free-falling frame $R^{\hat\kappa\hat\mu} \neq 0$ if there is non-trivial space-time curvature there. Finally, there's the question about finding how the β term affects the Einstein action, which follows from the variation of the action with $\delta g^{\mu\nu}$:

$$\frac{1}{8\pi G}(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{M\beta}}{\delta g^{\mu\nu}}.$$

Using expressions above,

$$-2\frac{1}{\sqrt{-g}}\frac{\delta S_{M\beta}}{\delta g^{\mu\nu}} = T^{EM}_{\mu\nu} + \beta F_{\kappa\lambda}F_{\rho\sigma}R^{\kappa\rho}g^{\lambda\sigma}g_{\mu\nu} - 2\beta F_{\mu\lambda}F_{\nu\sigma}R^{\lambda\sigma}$$

where we dropped the terms coming from $\delta R^{\kappa\rho}$ which, as mentioned earlier, need to be integrated by parts and then it can be checked that the result indeed vanishes.