2/8/17 Lecture 9 outline

• Recall from last time: de Sitter, anti- de Sitter, and Minkowski space are all conformally equivalent to the Einstein static universe.

de Sitter $\cosh(t/C) = 1/\cos t'$:

$$ds^2 = \frac{C^2}{\cos^2(t')}d\bar{s}^2, \qquad d\bar{s}^2 \equiv -(dt')^2 + d\chi^2 + \sin^2\chi d\Omega_2^2.$$

Here $-\pi/2 < t' < \pi/2$. Represent dS by a square, with t' on the y axis and $\chi \in [0, \pi]$ on the x axis. Spacelike slices are S^3s, so each point on the diagram is an S^2, except the edges $\chi = 0$ and $\chi = \pi$ are points, the North and South poles of the S^3. Diagonal lines are null rays. So a photon released at past infinity will get to an antipodal point on the sphere at future infinity. Note that points can have disconnected past or future light cones: the spherical spatial sections are expanding so light from one point cannot necessarily get to another.

Likewise anti-de Sitter: $\cosh\rho = 1/\cos\chi$

$$ds^2 = \frac{C^2}{\cos^2\chi}d\bar{s}^2,$$

where now $0 \leq \chi < \pi/2$ and t' is extended to run from $-\infty$ to $+\infty$.

Likewise, consider flat Minkowski space-time in spherical coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$. Now take $u = t - r$, $v = t + r$, $U = \arctan u$, $V = \arctan v$, $T = V + U$, $R = V - U$, to get a patch of the Einstein static universe, with $0 \leq R < \pi$ and $|T| + R < \pi$. Past time-like infinity i^- is $T = -\pi$, $R = 0$; future time-like infinity i^+ is $T = \pi$, $R = 0$, spatial infinity i^0 is $T = 0$, $R = \pi$, future and past null infinity, called scri $\mathcal{I}\pm$, are $T = \pm(\pi - R)$, for $0 < R < \pi$.

• Compare the causal structure of the three above cases (following Hawking and Ellis' The Large Scale Structure of Spacetime). In Minkowski, any future-directed timeline geodesic asymptotes to $i^\pm$ in the limit of $\tau \to \pm\infty$. So time-like geodesics start at i^- and end at i^+. Likewise, null geodes start on $\mathcal{I}^-$ and end on $\mathcal{I}^+$. These are only statements about geodesics, e.g. can write non-null time-like paths from $\mathcal{I}^-$ to $\mathcal{I}^+$. Cauchy surfaces, space-like and intersecting all time-like and null geodesics, end on i^0. Draw non-radial version of the surface, with light cones etc.

Now de Sitter space. Again, there are points on the space that are not joined by any geodesic. Recall square Penrose diagram: $\mathcal{I}^-$ is the lower boundary and $\mathcal{I}^+$ is the upper boundary. There are no boundaries on the left and right edges (draw them with dashed lines), they are coordinate boundaries corresponding to the poles of a S^2. The old steady-state universe (Hoyle) was the region $t' > \chi - \pi/2$, with $i^=$ at the lower left corner, $\chi = 0$, $t' = -\pi/2$, and i^0 at the upper-right corner, $t' = \pi/2$, $\chi = \pi$, and $\mathcal{I}^-$ the null diagonal connecting them.

de Sitter has a space like boundary for both time-like and null geodesics. Compare past light cone of a point in de Sitter vs Minkowski. The past null cone of a point p in de Sitter intersects $\mathcal{I}^-$ and leads to a horizon, with other particle's world lines outside the horizon. In Minkowski space, all particle geodesic world lines pass through the past light cone at any point p. So in de Sitter there are, at any given time, particles outside of the past light cone, whose effects cannot yet be seen. As observer's time increases, more and more particles come into the horizon. Likewise future light cones to $\mathcal{I}^+$, with some outside particles that will never be observable to the observer. Consider observer's trajectory from $\mathcal{I}^-$ to $\mathcal{I}^+$ and draw light cones for both. The region inside both is the maximal set of space-time events that O could have possibly influenced. Minkowski space-time does not have past or future horizons for geodesic observers (accelerated observers can have horizons). In the old steady state universe, there is a future horizon but no past horizon. Consider world-lines of observers O and Q in de Sitter, with Q initially outside of O's past null cone at some point p. Later Q's comes into O's future infinity null cone. So from O's perspective, Q can appear at some point, if looking through a telescope. Note that O thereafter never sees Q disappear, even though Q's world line can go out of O's future null cone. From O's point of view, an infinite amount of proper time passes as Q is approaching the point r where it goes outside of the future null cone. There is a redshift that approaches infinity for the light that O sees as Q approaches r. From Q's perspective, nothing special happens at r, they just pass right through in finite proper time. If Q is looking at O, they see O as having a huge redshift at a different time, and they likewise never see O disappear.

Finally anti-de Sitter space-time which has a time-like boundary with topology a hemisphere of S^3. The points i^- and i^+ are at $t' \to \mp\infty$ in the strip. Null rays can go from e.g. point p at $t' = -\pi/2$, $\chi = 0$ to $t' = 0$, $\chi = \pi/2$. Then the light ray can reflect off that boundary and go back to $\chi = 0$, reaching it at $t' = \pi/2$, point q. This is like the triangular region of Minkowski space, and any time-like geodesic starting at $t' = -\pi/2$ and $\chi = 0$ will stay within this region. So such time-like observers see a space-time causal structure that is no different than Minkowski space. Such time-like observers never reach the boundary at $\chi = \pi/2$, they refocus to point q in their infinite future. A god-like global observer, however, can see that the space-time continues for infinite t' in the past and future. There is no well-defined Cauchy surface, since any space-like surface will have future geodesics that do not intersect.

• If we assume the $n + \frac{1}{2}n(n-1)$ translational and space-time rotational Killing vectors, we have only the above three solutions of Einstein's equations. Now let's look for different solutions by assuming only translational and rotational symmetry in the space directions. The space directions are then a maximally symmetric space, and the time direction is allowed to differ, subject to Einstein's equations. These are the Robertson Walker space times.

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2,$$

where the 3d space $d\Sigma^2$ is maximally symmetric. Again, three possibilities: the 3d space can have $k = R_{3d}/6$ negative (open), positive (flat), or positive (closed). By a choice of coordinates,

$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

with $k = 0, 1, -1$. Or

$$d\Sigma^2 = d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with $f(\chi) = \sin\chi, \chi, \sinh\chi$, respectively, for $k = 1, 0, -1$. The χ ranges are $\chi_{k=0,-1} \in [0, \infty]$ and $\chi_{k=1} \in [0, \pi]$. The $k = 0, -1$ cases are infinite, topologically R^3, while the $k = 1$ case is closed, topologically S^3.

The symmetry of the RW space times require that the energy-momentum tensor be that of a perfect fluid: $T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu}$. Conservation of energy requires $\dot\rho/\rho = -3(1+w)\dot a/a$, where $w \equiv p/\rho$. For constant w this gives $\rho \sim a^{-3(1+w)}$. Recall e.g. that the null dominant energy condition conjecture is $|w| \leq 1$. Einstein's equations ($G_{\mu\nu}u^\mu u^\nu = 8\pi G\rho$ and $G_{\mu\nu}s^\mu s^\nu = 8\pi G p$) lead to the Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

For $a \neq 0$, the first equation can be obtained as the integral of the second one. It gives the constant of integration as being equal to the same constant k.

As found by Einstein, if ρ and p are non-negative, it is impossible to have a constant. He introduced a cosmological constant component, which has $p_\Lambda = -\rho_\Lambda$, to get constant a, which he later referred to as his greatest blunder (though he turned out to be right about $\Lambda \neq 0$).

It follows from the energy conservation equation that ρ decreases as the universe expands, and was higher in the past. As $a \to 0$, $\rho \to \infty$, so $a \to 0$ is a physical singularity, not just a harmless coordinate singularity. As $a \to 0$, space-time is singular, and Einstein's equations must break down before then, e.g. quantum effects must kick in. The past singularity "big bang" is there for all cases with $\rho + 3p > 0$. It can be evaded by e..g a positive Λ. For ρ and p non-negative, one can ask if the singularity can be evaded by a non-spherically symmetric configuration. Hawking proved in his PhD thesis that the singularity is still there, with fewer and fewer assumptions: singularity theorems. Will touch on them more later.

The Hubble parameter $H \equiv \dot{a}(t)$ is currently the Hubble constant $H_0 = H(t_0)$. $H(t_0)^{-1} \approx 9.78^{-1} h^{-1} \times 10^9$ years, with $h \approx .72$. Let $\rho_{crit} \equiv 3H_0^2.8\pi \equiv 1.99 \times 10^{-29} h^2 g/cm^3$. Define $\Omega_{m,r,v} \equiv \rho_{m,r,v}/\rho_{crit}$. Matter has $p_m \approx 0$, radiation (blackbody spectrum) has $p_r = \rho_r/3$, and vacuum CC has $p_v = -\rho_v$. If $\Omega = \Omega_v + \Omega_r + \Omega_v = 1$, then $k = 0$ and the universe is flat. This is what observation suggests to be the case in our universe: $\Omega_m \approx 4.6\%$, $\Omega_{d.m.} \approx 24\%$, $\Omega_v \approx 71.4\%$. The scaling of $\rho(t)$ is such that radiation dominated for $t \to 0$, then matter, and finally vacuum.

For $k = 0$ and $k = -1$, and $\rho > 0$, note that $\dot{a} > 0$ so the universe will expand forever. For any matter with $p > 0$, ρ must decrease as a increases at least as rapidly as a^{-3}, so $\rho a^2 \to 0$ as $a \to \infty$. So For $k = 0$ the expansion velocity $\dot{a} \to 0$ as $\tau \to \infty$, and for $k = -1$, $\dot{a} \to 1$. For $k = 1$, the universe cannot expand forever: eventually RHS wants to become negative, but the LHS is positive, so $a \leq a_{crit}$ and this happens for finite t. There is a bounce, where $a \to a_{crit}$ and then the universe re-contracts. A finite t after the big bang, $a \to 0$ again, in a big crunch. The spatially closed 3-sphere universe will only exist a finite span of time.

Get $\rho \sim a^{-n}$ with equation of state $w = \frac{1}{3}n - 1$. Matter has $n = 3$ (so $w = 0$), radiation has $n = 4$ (so $w = 1/3$), curvature has $n = 2$ (so $w = -1/3$), and vacuum has $n = 0$, so $w = -1$. For example, the Einstein static universe is a solution with $\rho_\Lambda = \frac{1}{2}\rho_M$; it is topologically $R \times S^3$.