2/6/17 Lecture 8 outline

• Maximally symmetric spaces: in n dimensions, have n translational Killing vectors and $\frac{1}{2}n(n-1)$ rotational Killing vectors. We do not assume any global transformations, like Lorentz transformations, merely the local existence of Killing vectors. This uniquely determines that $R_{\mu\nu\rho\sigma}$ has the form discussed last time, with $C_{\mu\nu\rho\sigma}=0$ and R a constant. Again, there are three cases, corresponding to R positive, negative, or zero.

Aside: Recall that Killing vectors K satisfy $\mathcal{L}_K g_{\mu\nu}=0$. This means that the Killing vector must satisfy

$$\nabla_\mu K_\nu + \nabla_\mu K_\nu = 0.$$

(Recall that the Lie derivative of a function is $\mathcal{L}_V f = V^\mu \partial_\mu f$, while that of tensors have additional terms, similar to the connection terms of covariant derivatives but instead involving V^μ, with same plus or minus signs depending on whether the indices are upper or lower, e.g. for a vector $\mathcal{L}_V U^\mu = [V,U]^\mu \equiv V^\nu\partial_\nu U^\mu - U^\nu \partial_\nu V^\mu$ and the Lie derivative of the metric along V is $\mathcal{L}_V g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} = \nabla_\mu V_\nu + \nabla_\mu V_\nu$). If K_μ is a Killing vector, then $K_\mu \frac{dx^\mu}{d\lambda}$ is conserved if $\frac{dx^\mu}{d\lambda}$ solves the geodesic equation. If p^μ is the 4-momentum of a test mass on a geodesic, $\nabla_{(\mu}K_{\nu)}$ implies conservation of $K_\nu p^\nu$: the geodesic equation gives $p^\lambda \nabla_\lambda p^\mu = 0$ so $p^\mu \nabla_\mu (K\cdot p) = p^\mu p^\nu \nabla_{(\mu}K_{\nu)} = 0$.

If e.g. the metric is independent of t then the Killing vector is $K^\mu = \delta^\mu_t = (1,0,0,0)$.

Using

$$(\nabla_\mu \nabla_\rho - \nabla_\rho \nabla \mu) V_\sigma = R^\lambda_{\sigma\rho\mu} V_\lambda,$$

and

$$R^\lambda_{(\sigma\rho\mu)} = 0,$$

find that a Killing vector has $K_{\mu;\rho;\sigma} = -R^\lambda_{\sigma\rho\mu}V_\lambda$.

For flat space-time, the Killing vectors are $K_\mu^{(\nu)} = \delta^\nu_\mu$ and $K_\mu^{[\nu\lambda]} = \delta^\nu_\mu x^\lambda - \delta^\lambda_\mu x^\nu$. One can similarly construct the Killing vectors of de Sitter or anti-de Sitter.

• Recall from last time: de Sitter, anti- de Sitter, and Minkowski space are all conformally equivalent to the Einstein static universe.

de Sitter $\cosh(t/C) = 1/\cos t'$:

$$ds^2 = \frac{C^2}{\cos^2(t')} d\bar{s}^2, \qquad d\bar{s}^2 \equiv -(dt')^2 + d\chi^2 + \sin^2\chi d\Omega_2^2.$$

Here $-\pi/2 < t' < \pi/2$. Represent dS by a square, with t' on the y axis and $\chi \in [0,\pi]$ on the x axis. Spacelike slices are S^3s, so each point on the diagram is an S^2, except the edges $\chi = 0$ and $\chi = \pi$ are points, the North and South poles of the S^3. Diagonal lines are null rays. So a photon released at past infinity will get to an antipodal point on the sphere at future infinity. Note that points can have disconnected past or future light cones: the spherical spatial sections are expanding so light from one point cannot necessarily get to another.

Likewise anti-de Sitter: $\cosh \rho = 1/\cos \chi$

$$ds^2 = \frac{C^2}{\cos^2 \chi} d\bar{s}^2,$$

where now $0 \leq \chi < \pi/2$ and t' is extended to run from $-\infty$ to $+\infty$.

Likewise, consider flat Minkowski space-time in spherical coordinates, $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$. Now take $u = t - r$, $v = t + r$, $U = \arctan u$, $V = \arctan v$, $T = V + U$, $R = V - U$, to get a patch of the Einstein static universe, with $0 \leq R < \pi$ and $|T| + R < \pi$. Past time-like infinity i^- is $T = -\pi$, $R = 0$; future time-like infinity i^+ is $T = \pi$, $R = 0$, spatial infinity i^0 is $T = 0$, $R = \pi$, future and past null infinity, called scri $\mathcal{I}\pm$, are $T = \pm(\pi - R)$, for $0 < R < \pi$.