1/30/17 Lecture 6 outline

Recall

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}; \qquad \text{so} \qquad R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \tag{1}$$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$G^{(1)}_{\mu\nu}(h) = -\tfrac{1}{2}\partial^2\bar{h}_{\mu\nu} + \tfrac{1}{2}\partial^\rho\partial_\mu\bar{h}_{\nu\rho} + \tfrac{1}{2}\partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \tfrac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}.$$

Choose an $x^\mu \to x^\mu + \xi^\mu$ gauge such that $\partial^\rho\bar{h}_{\rho\sigma} = 0$, which eliminates all but the first term in $G^{(1)}_{\mu\nu}$.

• Last time: Production of gravitational waves: want to solve $G^{(1)}_{\mu\nu} = 8\pi G T_{\mu\nu}$ which in the $\partial^\mu\bar{h}_{\mu\nu} = 0$ gauge choice becomes $-\partial^2\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}$. We know how to solve this equation, using $\vec{\nabla}^2(1/r) = -4\pi\delta^3(\vec{x})$, just like the Lienard-Wiechert potential in E&M :

$$\overline{h}_{\mu\nu}(t,\vec{x}) = 4G\int d^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|}T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y}).$$

Far away from the source, do a multipole expansion. The leading term is the quadrupole term:

$$h_{ij} \approx \frac{2G}{r}\frac{d^2I_{ij}}{dt^2}(t_r), \qquad I_{ij}(t) = \int d^3y y^i y^j T^{00}(t,\vec{y}).$$

Here is how to show it: the $\vec{y}$ integral above is over the past light cone of the space-time point x^μ. Let's F.T. $t \to \omega$:

$$\overline{h}_{\mu\nu}(\omega,\vec{x}) = 4G\int d^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|}T_{\mu\nu}(\omega,\vec{y})e^{i\omega|\vec{x}-\vec{y}|}.$$

The gauge condition gives $-i\omega\overline{h}_{0\mu}(\omega,\vec{x}) = \partial_i\overline{h}_{i\mu}(\omega,\vec{x})$, so we only need to solve for the $i,j \neq 0$ space components of the metric wave: $\mu \to i$ and $\nu \to j$. In the far zone, to leading order, we replace $|\vec{x}-\vec{y}| \to r$ and can take $e^{i\omega r}/r$ out of the integral. Now use

$$\int d^3\vec{y}T^{ij} = \int(\partial_k(T^{kj}y^i) - \partial_k T^{kj}y^i) = -i\omega\int T^{0(j}y^{i)} =$$

$$= \tfrac{1}{2}\int(\partial_k(T^{0k}y^iy^j) - \partial_k T^{0k}y^iy^j) = -\tfrac{1}{2}\omega^2\int T^{00}y^iy^jd^3\vec{y}$$

(using conservation of $T^{\mu\nu}$ in leading, approximately flat-space form, and dropping surface terms since we take the source to be compact). So the leading term is quadrupole radiation, unlike E&M where the leading term is dipole radiation. There is no dipole radiation here because conservation of momentum implies that the dipole term is a constant in time.

E.g. two stars of mass M, separated by distance $2R$ in the weak field, non-relativistic limit have $I_{ij} \sim MR^2$ and $d^2 I_{ij}/dt^2 \sim \Omega^2 MR^2$ with $\Omega = 2\pi/T = v/R = \sqrt{GM/4R^3}$.

• How much power is carried away in the gravitational radiation of the quadrupole? Recall from last time we can write

$$G^{(1)}_{\mu\nu} = 8\pi G(T_{\mu\nu} + t_{\mu\nu}), \qquad t_{\mu\nu} \equiv \frac{1}{8\pi G}(G_{\mu\nu} - G^{(1)}_{\mu\nu})$$

and we can sort-of interpret $t_{\mu\nu}$ as the energy-momentum of the gravitational fields. To leading, quadratic order in h:

$$t_{\mu\nu} \approx \frac{1}{8\pi G}\left(-\tfrac{1}{2}h_{\mu\nu}\eta^{\rho\lambda}R^{(1)}_{\rho\lambda} + \tfrac{1}{2}\eta_{\mu\nu}h^{\rho\lambda}R^{(1)}_{\rho\lambda} + R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\lambda}R^{(2)}_{\rho\lambda}\right).$$

As we discussed last time, this is not gauge invariant so not fully kosher. Nevertheless, it has some merit. If we integrate it, the gauge variations drop out to leading order in small h perturbation theory around flat space, so this is a sensible way to compute the power radiated. Since $G^{(1)}_{\mu\nu}$ satisfies the leading order Einstein's equations, we have

$$t_{\mu\nu} \approx \frac{1}{8\pi G}(R^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\eta^{\rho\lambda}R^{(2)}_{\rho\lambda}).$$

Upshot: the radiated power is related to the derivative of h_{ij} squared. As above, h_{ij} is related to a 2nd time derivative of the quadrupole moment. So the power is related to the third derivative of the quadrupole squared (the details and indices are tedious to work out):

$$P = \frac{G}{5c^9}\frac{d^3 I_{ij}(t_r)}{dt^3}\frac{d^3 I^{ij}(t_r)}{dt^3}.$$

(check units and put back in c's: $[G] = M^{-1}(L/T)^3 T$, $[\frac{d^3 I}{dt^3}] = M(L/T)^4 T^{-1}$, $P = M(L/T)^2 T^{-1}$). See e.g. Weinberg ch 10.5 for details, e.g. the factor of 1/5 is related to the integral of four unit vectors over all solid angle.

Example (details in Weinberg ch 10.5): Weber's attempt to directly observe gravitational radiation from a sound vibration in an large aluminum cylinder. The density in the cylinder is $\rho = \rho_0 + \rho_1$, where ρ_1 is the vibrating sound wave fluctuation: $\rho_1 = \epsilon\rho_0 \sin(kz)\cos(\omega t)$. There is a quadrupole term $I_{zz} = A\int_0^L \rho_1 z^2 dz$. Find energy loss:

$$\Gamma_{grav} = \frac{P}{E} = \frac{64 G M v_s^4}{15 L^2 c^5},$$

where v_s is the sound velocity. Plug in Weber's values: $L = 1.53m$, $v_s = 5.1 \times 10^3 m/s$, $M = 1.4 \times 10^3 kg$, get $\Gamma_{grav} = 4.7 \times 10^{-35} s^{-1}$. It proved to be too small for him to eliminate all systematics and accurately and precisely measure. Instead, tried to use the bars as detectors of gravity waves produced in space. Found some possible effects, but could not eliminate the systematics to show they were actually gravity waves; so did not succeed.

Another example (again, Weinberg ch 10.5). Gravitational radiation of a large, rotating body. If rotation frequency is Ω, the radiation has frequency peaked at 2Ω (since we square h). The result is

$$P(2\Omega) = \frac{32 G \Omega^5 I^2 e^2}{5 c^5},$$

where $I = I_{11} + I_{22}$ and $e = (I_{11} - I_{22})/I$, where the rotation is around the 3 axis. Only radiates if not axially symmetric around axis of rotation. Apply e.g. to the rotation of Jupiter around the sun, take $e = 1$, $I = mr^2$, $\Omega = 1.68 \times 10^{-8} s^{-1}$, $m = 1.9 \times 10^{27} kg$, $r = 7.79 \times 10^{11} m$, get $P \approx 5.3 kW$, tiny.

Consider two stars, each of mass M, separated by distance $2R$, rotating with frequency Ω. Evaluate $I_{xx} = 2MR^2 \cos^2(\Omega t)$, $I_{xy} = MR^2 \sin(2\Omega t)$, $I_{yy} = 2MR^2 \sin^2(\Omega t)$. Let $\Omega = 2\pi/T$ with T the period and use Kepler's law $V^2/R = GM/(2R)^2$ to get $R = (GMT^2/16\pi^2)^{1/3}$. Average over a period. Get

$$\langle P \rangle = \frac{128}{5c^5} G M^2 R^4 \Omega^6 = \frac{128}{5} 4^{1/3} \frac{c^5}{G} \left(\frac{\pi G M}{c^3 T} \right)^{10/3}.$$

$$= 1.9 \times 10^{26} \left(\frac{M}{M_{sun}} \frac{1h}{T} \right)^{10/3} W.$$

By comparison, the sun radiates electromagnetic radiation $3.9 \times 10^{26} W$ and a large galaxy about $10^{37} W$ and a bright gamma-ray burst about $10^{45} W$. The LIGO event of Sept 14, 2015 was estimated to have a peak gravitational power radiation of $3.6 \times 10^{49} W$, which for that brief period was greater than all light radiated by all stars in the visible universe. The approximations that led to the above formula are of course invalid for getting the power radiated by black-hole mergers. The event was interpreted as an inspiral of a 35 solar mass black hole and a 30 solar mass black hole, resulting in a 62 solar mass black hole. The mass and rotational energy difference was radiated primarily in gravity waves. The GR analysis could not be done analytically; it was done numerically on computers.

• Consider isotropic and homogeneous space times: looks the same in all directions and the same under translations. A maximally symmetric space-time of dimension n has

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}),$$

with R the Ricci scalar that is constant over the space-time. The Weyl tensor for these spaces is $C_{\rho\sigma\mu\nu} = 0$. There are three possibilities: $R = 0$: flat; $R > 0$: de Sitter; $R < 0$: anti-de Sitter. These are solutions of Einstein's equations for $T_{\mu\nu} \sim g_{\mu\nu}$, with zero, positive, and negative CC respectively. For $n = 4$ we have $R_{\mu\nu} = 3\kappa g_{\mu\nu}$, where $R = 12\kappa$ and Einstein's equations are satisfied if $\rho = -p = 3\kappa/8\pi G$.

One way to get de Sitter space is to start in 5d, with $ds_5^2 = -du^2+dx^2+dy^2+dz^2+dw^2$ and restrict to a hyperboloid $-u^2 + x^2 + y^2 + z^2 + w^2 = C^2$, where C is the de Sitter radius. By taking $u = C\sinh(t/C)$, and $w, z, y, z \sim C\cosh(t/C)$ times S^3 coordinates, the metric is

$$ds^2 = -dt^2 + C^2\cosh^2(t/C)d\Omega_3^2,$$

where $d\Omega_3^2$ is the solid angle on an S^3. These are geodesically complete coordinates, so the topology is $R \times S^3$.

Likewise anti-de Sitter starts with $ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2$ and restricts to a hyperboloid $u^2 + v^2 - x^2 - y^2 - z^2 = C^2$. Write $u = C\sin(t')\cosh\rho$, $v = C\cos(t')\cosh\rho$, and $x, y, z \sim C\sinh\rho$ times S^2 coordinates, gives

$$ds^2 = C^2(-\cosh^2\rho dt'^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2).$$

The t' coordinate is a closed time-like curve, which is bad, so consider the covering space where t' is not identified with $t' + 2\pi$.

• Recall Friedmann Robertson Walker:

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2,$$

and consider case where the 3d space $d\Sigma^2$ is maximally symmetric. Again, three possibilities: the 3d space can have $k = R_{3d}/6$ negative (open), positive (flat), or positive (closed). By a choice of coordinates,

$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

with $k = 0, 1, -1$. These spaces solve Einstein's equations for a fluid $T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu}$. Conservation of energy requires $\dot\rho/\rho = -3(1+w)\dot a/a$, where $w \equiv p/\rho$. For constant

w this gives $\rho \sim a^{-3(1+w)}$. Recall e.g. that the null dominant energy condition conjecture is $|w| \leq 1$. Einstein's equations lead to the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

Get $\rho \sim a^{-n}$ with equation of state $w = \frac{1}{3}n - 1$. Matter has $n = 3$ (so $w = 0$), radiation has $n = 4$ (so $w = 1/3$), curvature has $n = 2$ (so $w = -1/3$), and vacuum has $n = 0$, so $w = -1$. For example, the Einstein static universe had $\rho_\Lambda = \frac{1}{2}\rho_M$; it is topologically $R \times S^3$.

• Recall Schwarzschild solution, e.g. for a $T^{\mu\nu}$ a spherically symmetric, static, delta function at the origin. Away from the origin, $T^{\mu\nu} = 0$, so we solve Einstein's equations in vacuum, $R_{\mu\nu} = 0$. There is Birkhoff's theorem, that there is a unique vacuum solution with spherical symmetry, and it turns out to be static. If we take[1]

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \tag{2}$$

can compute $R_{\mu\nu}$ and see that it vanishes only if $\alpha = -\beta$ and $\partial_r(re^{2\alpha}) = 1$, which gives the Schwarzschild solution, $e^{2\alpha} = 1 - R_s/r$. Recall that we know from the Newtonian limit that $h_{00} = 2\Phi$, so $R_s = 2GM$. The Ricci tensor vanishes for Schwarzschild, but the Riemann tensor does not. Write out some example components, e.g. $R^r_{\phi r\phi} = re^{-2\beta}\sin^2\theta\partial_r\beta$, $R^t_{\theta t\theta} = -GM/r$, etc. The non-zero Riemann tensor will give e.g. the correct focusing of nearby geodesics,

$$\frac{D^2}{d\lambda^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\delta x^\sigma.$$

Using the full metric, can explore this beyond the linearized limit discussed above. Can show

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}.$$

We see that $r = 0$ is really a singularity whereas $r = R_s$ is not a real singularity.

• Since

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

[1] a $e^{2\gamma(r)}$ factor in front of the $r^2 d\Omega^2$ term could be eliminated by a redefinition of $r \to e^\gamma r$

is static and rotationally invariant, there are four Killing vectors corresponding to H and $\vec{L}$.

Recall that Killing vectors K satisfy $\mathcal{L}_K g_{\mu\nu} = 0$. (Recall that the Lie derivative of a function is $\mathcal{L}_V f = V^\mu \partial_\mu f$, while that of tensors have additional terms, similar to the connection terms of covariant derivatives but instead involving V^μ, with same plus or minus signs depending on whether the indices are upper or lower, e.g. for a vector $\mathcal{L}_V U^\mu = [V,U]^\mu \equiv V^\nu \partial_\nu U^\mu - U^\nu \partial_\nu V^\mu$ and the Lie derivative of the metric along V is $\mathcal{L}_V g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} = \nabla_\mu V_\nu + \nabla_\mu V_\nu$). If K_μ is a Killing vector, then $K_\mu \frac{dx^\mu}{d\lambda}$ is conserved if $\frac{dx^\mu}{d\lambda}$ solves the geodesic equation. If p^μ is the 4-momentum of a test mass on a geodesic, $\nabla_{(\mu} K_{\nu)}$ implies conservation of $K_\nu p^\nu$: the geodesic equation gives $p^\lambda \nabla_\lambda p^\mu = 0$ so $p^\mu \nabla_\mu (K \cdot p) = p^\mu p^\nu \nabla_{(\mu} K_{\nu)} = 0$.

If e.g. the metric is independent of t then the Killing vector is $K^\mu = \delta^\mu_t = (1,0,0,0)$.

So for Schwarzschild we have

$$H \to \partial_t \to K_\mu = (-(1 - \frac{2GM}{r}), 0, 0, 0).$$

$$L_z \to \partial\phi \to L_\mu = (0, 0, 0, r^2 \sin^2\theta).$$

Taking $\theta = \pi/2$, the conserved quantities are

$$E = (1 - \frac{2GM}{r})\frac{dt}{d\lambda}, \qquad L = r^2 \frac{d\phi}{d\lambda}.$$

(For massive particles, it is actually L_z/m.) The orbits can be written as

$$\tfrac{1}{2}(\frac{dr}{d\lambda})^2 + V_{eff}(r) = \tfrac{1}{2}E^2,$$

$$V_{eff}(r) = \tfrac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3},$$

with $\epsilon \equiv -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ so $\epsilon = 1$ for a massive particle with $\lambda = \tau$ and $\epsilon = 0$ for a massless particle.