1/24/17 Lecture 5 outline

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}; \qquad \text{so} \qquad R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \tag{1}$$

• Last time: take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and treat $h_{\mu\nu}$ as a small perturbation and linearize,

$$R^{(1)}_{\mu\nu} \approx \tfrac{1}{2}(\partial^\sigma \partial_\nu h_{\mu\sigma} + \partial^\sigma \partial_\mu h_{\nu\sigma} - \partial_\mu \partial_\nu h - \partial^2 h_{\mu\nu}),$$

$$R^{(1)} \approx \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h.$$

Plug in to get $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$. It looks slightly nicer if expressed in terms of $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$:

$$G^{(1)}_{\mu\nu}(h) = -\tfrac{1}{2}\partial^2 \bar{h}_{\mu\nu} + \tfrac{1}{2}\partial^\rho \partial_\mu \bar{h}_{\nu\rho} + \tfrac{1}{2}\partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \tfrac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}.$$

As discussed last time, we can choose an $x^\mu \to x^\mu + \xi^\mu$ gauge such that $\partial^\rho \bar{h}_{\rho\sigma} = 0$, which eliminates all but the first term in $G^{(1)}_{\mu\nu}$.

• Last time: gravity waves in empty space. Take $T_{\mu\nu} = 0$ in Einstein's equations, and linearize them to get $\partial^2 s_{ij} = 0$. Call $h^{TT}_{\mu\nu} = 2s_{ij}$ for the i, j components and zero otherwise. Write a plane wave solution, $h^{TT}_{\mu\nu} = e_{\mu\nu}e^{ikx} + c.c.$, which solves the wave equation for $k^2 = 0$: the graviton is massless. To keep it transverse (eliminate gauge dof), need $k^\mu e_{\mu\nu} = 0$. Taking $k^\mu = (\omega, 0, 0, \omega)$, find, 2 independent polarization components, $e_{11} = h_+$ and $e_{12} = h_X$. A ring of particles in the $x - y$ plane will oscillate in a $+$ shape in reaction to a gravitational wave with $h_+ \neq 0$, and $h_X = 0$. A gravitational wave with $h_X \neq 0$ and $h_+ = 0$ will cause them to oscillate in a X pattern. Can define $h_{R,L} = (h_+ \pm ih_X)/\sqrt{2}$ circular polarizations.

• Aside on currents and the energy momentum tensor. In E&M, the charge density current of a bunch of point charges is $J^\mu = g^{-1/2}\sum_a q_a \int \delta^4(x - x_a)dx^\alpha_a$, where the integral is over the particle's world-line and $g^{-1/2}\delta^4(x)$ is coordinate invariant, just as $g^{1/2}d^4x$ is. Likewise, the energy momentum tensor of a system of point particles is $T^{\mu\nu} = g^{-1/2}\sum_a m_a \int p^\mu_a dx^\nu_a \delta^4(x - x_a)$. For massive particles $p^\mu_a = m dx^\mu_a/d\tau$. In the flat-space, non-relativistic limit, to order v^0, we have $T^{00} \approx \sum_a m_a \delta^3(\vec{x} - \vec{x}_a) = \rho$ with all other components zero. To order v, the non-zero components are $T^{0i} \approx \sum_a m_a v^i_a \delta^3(\vec{x} - \vec{x}_a)$, with all other components zero (this is enough for one of the HW questions).

• Beyond the linearized approximation. Recall that in E&M we have $T^{\mu\nu}_{total} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{field}$ and translation invariance implies $\partial_\mu T^{\mu\nu}_{total} = 0$, so $P^\mu_{total} = \int_V d^3\vec{x}T^{0\mu}$ is conserved:

it can only change if there is a flux of energy-momentum through the boundary ∂V. The matter and field energy and momentum are of course not separately conserved, since energy and momentum can be exchanged between the matter and $\vec{E}$ and $\vec{B}$ fields: $\partial_\mu T^{\mu\nu}_{field} = -\frac{1}{c}F^{\nu\lambda}J_\lambda$, $\partial_\mu T^{\mu\nu}_{matter} = +\frac{1}{c}F^{\nu\lambda}J_\lambda$.

We can get insight into the analogous issues for the energy and momentum of gravity by going beyond the linearized approximation. Define $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$, where now we will not impose that $h_{\mu\nu} = h^{(1)}_{\mu\nu}$ and consider Einstein's equations re-written as

$$G^{(1)}_{\mu\nu} = 8\pi G(T_{\mu\nu} + t_{\mu\nu}), \qquad t_{\mu\nu} \equiv \frac{1}{8\pi G}(G_{\mu\nu} - G^{(1)}_{\mu\nu}).$$

It looks trivial. The idea will be to try to interpret $T_{\mu\nu}$ as the matter contribution, and $t_{\mu\nu}$, as sort-of like an energy-momentum tensor for the gravitational field, so the thing on the RHS of the first equation is like a total energy-momentum tensor. Note that $G^{(1)}_{\mu\nu}$ satisfies the linearized Bianchi identity, $\partial^\mu G^{(1)}_{\mu\nu} = 0$ (i.e. ordinary not covariant derivatives) so $\tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu}$ satisfies $\partial^\mu \tau_{\mu\nu} = 0$, again without the covariant derivatives. This all looks bad for general covariance but good for conservation of $\tau_{\mu\nu}$ without the extra contributions from covariant derivatives. This all does not literally make sense (it is not gauge invariant); it can nevertheless be used to define a well-defined energy $P^\mu = \int_\Sigma \tau^{0\mu} d^3x$ where Σ is a space like surface. Likewise for the angular momentum: $J^{\mu\nu} = \int d^3x(x^\mu \tau^{\nu 0} - x^\nu \tau^{\mu 0})$.

See Wald 4.4 for more details.

In particular, if we consider Einstein's equations to 2nd order in $h_{\mu\nu}$, in vacuum. Need to work out $R^{(2)}_{\mu\nu}$ to order h^2. To satisfy Einstein's equations, need to correct metric, $h_{\mu\nu} = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \ldots$, with $G^{(2)}_{\mu\nu}[h^{(1)}] + G_{\mu\nu}[h^{(2)}] = 0$. Write this as $G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi t_{\mu\nu} \equiv -G^{(2)}_{\mu\nu}[h^{(1)}]$. This $t_{\mu\nu}$ looks roughly analogous to $T_{\mu\nu,field}$ in E&M, with $A_\rho \to h_{\rho\sigma}$.

• Energy and momentum of gravitational plane waves: plug $h^{TT}_{\mu\nu}$ into $t_{\mu\nu}$. The expression looks complicated but simplifies if we space-time average to eliminate all terms like $e^{\pm 2ikx}$. E.g. $\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}k_\mu k_\nu (e^{\lambda\rho *} e_{\lambda\rho} - \frac{1}{2}|e^\lambda_\lambda|^2)$ and $\langle t_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{8\pi G}(|e_{11}|^2 + |e_{12}|^2)$.

• Production of gravitational waves: want to solve $G^{(1)}_{\mu\nu} = 8\pi G T_{\mu\nu}$ which in the $\partial^\mu \bar{h}_{\mu\nu} = 0$ gauge choice becomes $-\partial^2 \bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}$. We know how to solve this equation, using $\vec{\nabla}^2(1/r) = -4\pi\delta^3(\vec{x})$, just like the Lienard-Wiechert potential in E&M :

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int d^3\vec{y} \frac{1}{|\vec{x} - \vec{y}|} T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y}).$$

Far away from the source, do a multipole expansion. The leading term is the quadrupole term:

$$h_{ij} \approx \frac{2G}{r}\frac{d^2 I_{ij}}{dt^2}(t_r), \qquad I_{ij}(t) = \int d^3y y^i y^j T^{00}(t, \vec{y}).$$

E.g. two stars of mass M, separated by distance $2R$ in the weak field, non-relativistic limit have $I_{ij} \sim MR^2$ and $d^2 I_{ij}/dt^2 \sim \Omega^2 MR^2$ with $\Omega = 2\pi/T = v/R = \sqrt{GM/4R^3}$.