1/23/17 Lecture 4 outline

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}; \qquad \text{so} \qquad R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \tag{1}$$

• Consider weak field limit, so $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small, and linearize in $h_{\mu\nu}$ e.g. $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$ and

$$\Gamma^\rho_{\mu\nu} \approx \tfrac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}).$$

And we can drop the $\Gamma\Gamma$ terms in the Riemann tensor, so

$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - [\mu \leftrightarrow \nu] \approx \partial_\nu \Gamma^\sigma - [\mu \leftrightarrow \nu],$$

$$R_{\mu\nu\rho\sigma} \approx \tfrac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - [\mu \leftrightarrow \nu]).$$

$$R_{\mu\nu} \approx \tfrac{1}{2}(\partial^\sigma\partial_\nu h_{\mu\sigma} + \partial^\sigma\partial_\mu h_{\nu\sigma} - \partial_\mu\partial_\nu h - \partial^2 h_{\mu\nu}),$$

$$R \approx \partial_\mu\partial_\nu h^{\mu\nu} - \partial^2 h.$$

Plug in to get $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$. It looks slightly nicer if expressed in terms of $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

Gauge freedom: two metrics are equivalent if they differ by a diffeomorphism (an invertible map, i.e. a coordinate change). An infinitesimal diffeomorphism is generated by $x^\mu \to x^\mu + \xi^\mu(x)$, $g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\mu\xi_\nu$. So $h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$ is the linearized limit gauge transformation. The linearized Riemann tensor is invariant: $\delta R_{\mu\nu\rho\sigma} = 0$ thanks to cancellation among the terms from the antisymmetrized indices. It is conventional to write $h_{00} \equiv -2\Phi$, $h_{0i} \equiv w_i$, $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$, with $s_{ij} \equiv \frac{1}{2}(h_{ij} - \frac{1}{3}\delta_{ij}h)$ and $h \equiv \delta^{ij}h_{ij} \equiv -6\Psi$. Defining also $G^i \equiv -\partial_i\Phi - \partial_0 w_i$ and $H^i \equiv \epsilon^{ijk}\partial_j w_k$, the geodesic equation gives

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0 \to \frac{dp^\mu}{dt} = \frac{d\lambda}{dt}\frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E}$$

and the spatial terms can be

$$\frac{dp^i}{dt} = E\left(G^i + (\vec{v}\times\vec{H})^i - 2(\partial_0 h_{ij})v^j - (\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk})v^j v^k\right)$$

and the first two terms remind one of the Lorentz force law of E&M.

By taking $\partial^2\xi_\mu = -\partial^\sigma\bar{h}_{\sigma\mu}$, we can set $\partial^\nu\bar{h}_{\mu\nu} = 0$. This is the analog of Lorentz gauge in Maxwell's equations.

Consider first the static, Newtonian limit: $T_{\mu\nu} \approx diag(\rho, 0, 0, 0)$, so

$$R_{00} = 8\pi G(T_{00} - \tfrac{1}{2}Tg_{00}) \approx 4\pi G\rho.$$

Using the gauge transformation, pick gauge such that

$$0 = g^{\mu\nu}\Gamma^{\rho}_{\mu\nu} \to \partial_\mu h^\mu_\lambda - \tfrac{1}{2}\partial_\lambda h = 0$$

and then the linearized Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ become

$$\partial^2 \overline{h}_{\mu\nu} \approx -16\pi G T_{\mu\nu}, \qquad \overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h.$$

If $T_{00} = \rho$ is the only non-negligible component of $T_{\mu\nu}$ get $\overline{h}_{i0} \approx \overline{h}_{ij} \approx 0$ and then

$$ds^2 \approx -(1+2\Phi)dt^2 + (1-2\Phi)d\vec{x}\cdot d\vec{x}.$$

use $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$, in the static Newtonian limit to get $R_{00} \approx R^i_{0i0} \approx -\frac{1}{2}\nabla^2 h_{00} = \nabla^2\Phi$, so the Newtonian limit checks: $\nabla^2\Phi = 4\pi\rho$.

In the weak field, Newtonian limit: $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \approx 1$, so geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} \approx -\Gamma^\mu_{00} = \tfrac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \approx -\partial^\mu\Phi.$$

Get the Newtonian limit, with e.g. $\Phi = -GM/r$.

Geodesic deviation:
$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\sigma.$$

Here we get

$$\frac{D^2}{D\tau^2}\delta x^i \approx R^i_{00j}\delta x^j = -R^i_{0j0}\delta x^j \approx -\partial^i\partial_j\Phi\delta x^j = -\delta(\partial^i\Phi),$$

fitting with the Newtonian picture that $\vec{a} = -\vec{\nabla}\Phi$.

- Linearized Einstein equations continued (for general $T_{\mu\nu}$). Names for components, $h_{00} = -2\Phi$, $h_{0i} = w_i$, and $h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}$. Then $\Gamma^0_{00} = \partial_0\Phi$, etc. The geodesic equation (taking $\lambda = \tau/m$ for massive particles)

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0$$

then gives, using $p^0 = dt/d\lambda = E$ and $p^i = Ev^i$,

$$\frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\sigma}\frac{p^\rho p^\sigma}{E},$$

or in components

$$\frac{dE}{dt} = -E(\partial_0\Phi + 2(\partial_k\Phi)v^k - (\partial_{(j}w_{k)} - \tfrac{1}{2}\partial_0 h_{jk})v^j v^k),$$

(giving energy exchange between the particle and gravity) and

$$\frac{dp^i}{dt} = E[G^i + (\vec{v}\times H)^i - 2(\partial_0 h_{ij})v^j - (\partial_{(j}h_{k)i} - \tfrac{1}{2}\partial_i h_{jk})v^j v^k]$$

where $G^i \equiv -\partial_i\Phi - \partial_0 w_i$, and $H^i \equiv \epsilon^{ijk}\partial_j w_k$.

• Coordinate transformation, $\delta h_{\mu\nu} = \partial_{(\mu}\epsilon_{\nu)}$ similar to gauge transformations in E&M. Can pick convenient gauges, e.g. set $\Phi = w^i = 0$. The scalars Φ and Ψ are would-be scalars, but aren't physical. Neither is the would-be spin 1 component w_i. The only physical dof are the spin $s = 2$ quadrupole components s_{ij}. This looks like $2s+1 = 5$ components, but there's still more gauge redundancy. Actually, only 2 independent physical polarizations. Counting: $h_{\mu\nu}$ has 10 polarizations, minus 4 for $\delta x^\mu = \epsilon^\mu(x)$ symmetry, minus another 4 for the longitudinal condition, gives 2. Gauge symmetry "cuts twice," like in E&M where we have $4-1-1=2$, here we have $10-4-4=2$.

• Gravity waves in empty space. Take $T_{\mu\nu} = 0$ in Einstein's equations, and linearize them to get $\partial^2 s_{ij} = 0$. Call $h^{TT}_{\mu\nu} = 2s_{ij}$ for the i, j components and zero otherwise. Write a plane wave solution, $h^{TT}_{\mu\nu} = C_{\mu\nu}e^{ikx}$, which solves the wave equation for $k^2 = 0$: the graviton is massless. To keep it transverse (eliminate gauge dof), need $k^\mu C_{\mu\nu} = 0$. Taking $k^\mu = (\omega, 0, 0, \omega)$, find, 2 independent polarization components, $C_{11} = h_+$ and $C_{12} = h_X$. A ring of particles in the $x-y$ plane will oscillate in a $+$ shape in reaction to a gravitational wave with $h_+ \neq 0$, and $h_X = 0$. A gravitational wave with $h_X \neq 0$ and $h_+ = 0$ will cause them to oscillate in a X pattern. Can define $h_{R,L} = (h_+ \pm ih_X)/\sqrt{2}$ circular polarizations.