1/18/17 Lecture 3 outline

- Quote some equations from last times:

$$S[g,X] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{\text{everything else}}[\eta, X, \partial_\mu X]\big|_{\eta\to g, \partial\to\nabla}. \tag{1}$$

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2}$$

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \tag{3}$$

$$T^{cc}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}. \tag{4}$$

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{else}_{\mu\nu}. \tag{5}$$

$T_{\mu\nu} = -\rho_\Lambda g_{\mu\nu} + T^{\text{else}}_{\mu\nu}$. Supernovae observations (1998) + other tests

$$\rho_\Lambda = (1.148 \pm 0.11) \times 10^{-123}$$

- Weyl tensor:

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(d-1)(d-2)}g_{\rho[\mu}g_{\nu]\sigma}R.$$

Using the Bianchi identity $\nabla_{[\lambda}R_{\rho\sigma]\mu\nu}$ can show (taking $d=4$)

$$\nabla^\rho C_{\rho\sigma\mu\nu} = \nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{6}g_{\sigma[\mu}\nabla_{\nu]}R$$

and then Einstein's equations gives a differential equation for $C_{\rho\sigma\mu\nu}$

$$\nabla^\rho C_{\rho\sigma\mu\nu} = 8\pi G(\nabla_{[\mu}T_{\nu]\sigma} + \frac{1}{3}g_{\sigma[\mu}\nabla_{\nu]}T)$$

Analogous to $\nabla_\mu F^{\nu\mu} = J^\nu$: various solutions depending on boundary conditions, including propagating waves etc.

- Expect that some $g_{\mu\nu}$ are unphysical and should not arise, e.g. expect no closed, time-like curves. But any crazy $g_{\mu\nu}(x)$ will solve Einstein's equations, by definition, for appropriate $T_{\mu\nu}$. So anticipate that unphysical $g_{\mu\nu}(x)$ are avoided by some physical conditions on $T_{\mu\nu}$. Several possibilities have been proposed, e.g. the Weak Energy Condition states that $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all time-like t^μ. E.g. consider a perfect fluid:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$$

where U^μ is the fluid's 4-velocity. Then the WEC requires $T_{\mu\nu}U^\mu U^\nu \geq 0$ and $T_{\mu\nu}n^\mu n^\nu \geq 0$, where n^μ is a null vector. Get $\rho \geq 0$ and $\rho + p \geq 0$. An even weaker condition is the Null Energy Condition, which only requires $T_{\mu\nu}n^\mu n^\nu \geq 0$ for null vectors, so it only requires $\rho + p \geq 0$, with no separate condition on ρ. The Strong Energy Condition requires $T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\nu_\nu t^\rho t_\rho$ for all time-like t^μ, which implies $\rho + p \geq 0$ and $\rho + 3p \geq 0$ (but does not require $\rho \geq 0$ so it does not imply the WEC). Equation of state parameter $w \equiv p/\rho$.

- Geometric vectors and forms. Exterior derivative of forms.