1/11/17 Lecture 2 outline

• Introduction, continued. Last time: Einstein's equations follow from varying the Einstein-Hilbert action with respect to $\delta g_{\mu\nu}$:

$$S = \int d^d x\sqrt{|g|}\left[\frac{1}{16\pi G}R + \mathcal{L}_{\rm matter}(\eta\to g, \partial_\mu \to \nabla_\mu) + \mathcal{L}_{\rm other}\right].$$

The $g_{\mu\nu}$ Euler Lagrange equations then give

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad T_{\mu\nu} = -2\frac{1}{\sqrt{|g|}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}.$$

The last expression for $T_{\mu\nu}$ is equivalent to that found via Noether's procedure. The $\mathcal{L}_{\rm other}$ possible terms will lead to variants of (deviations from) Einstein's equations. Fill in some details here.

• Continue from last time.

$$S[g,X] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + S_{\rm everything\ else}[\eta, X, \partial_\mu X]\big|_{\eta\to g, \partial\to\nabla}. \tag{1}$$

(Last time we left the coefficient of the gravity term as α/G, and left it that we'd determine the coefficient by checking agreement with the Newtonian limit. Let's now just cut to the chase and write the answer, and check that it's right.)

The variation of (1) with $g_{\mu\nu}\to g_{\mu\nu}+\delta g^{\mu\nu}$ gives

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}R_{\mu\nu}g^{\mu\nu} + 8\pi G T_{\mu\nu} \tag{2}$$

where we used the relation discussed last time,

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{everything\ else}}{\delta g^{\mu\nu}}. \tag{3}$$

A way to relate this to the usual notion is to consider translations $x^{\mu'} = x^\mu + a^\mu$ and then, linearizing in small a^μ, get $\delta g^{\mu\nu}\approx a^{\mu;\nu} + a^{\nu;\mu}$, where the ; means covariant derivative. Then get $\delta S = \int d^4x T_{\mu\nu}a^{\mu;\nu}$. Note this shows $T_{\mu\nu}^{;\nu} = 0$, covariant energy-momentum conservation. For a macroscopic body, get $T_{\mu\nu} = (p+\rho)u_\mu u_\nu + pg_{\mu\nu}$.

Now use $g = e^{-{\rm Tr}\ln g^{\mu\nu}}$ to get $\delta g = -g g_{\mu\nu}\delta g^{\mu\nu}$, so $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Also, get that $\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu} - (\lambda\leftrightarrow\nu)$ and then that the $g^{\mu\nu}\delta R_{\mu\nu}$ term contributes only total covariant derivative terms, $\nabla_\rho\nabla_\sigma(-\delta g^{\rho\sigma} + g^{\rho\sigma}g_{\alpha\beta}\delta g^{\alpha\beta})$, that can be dropped.

So we have, finally, Einstein's equations:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4}$$

As a first check that things are good, note that energy-momentum conservation $\nabla^\mu T_{\mu\nu} = 0$ is compatible with this equation, thanks to the Bianchi identity discussed last week, $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$.

Let's rewrite (4) another convenient way. Get $R - 2R = -R = 8\pi G T$, $T \equiv T^\mu_\mu$, so

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}). \tag{5}$$

• Cosmological constant. There can be a constant term in S_{else} in (1),

$$S_{everything\ else} = S_{else} + \int d^4x\sqrt{-g}\left(\frac{-\Lambda}{16\pi G}\right). \tag{6}$$

The cc contributes to (3),

$$T^{cc}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}. \tag{7}$$

So $\rho_{vac} = -p_{vac} = \Lambda/8\pi G$ (sometimes this is called the c.c.). Einstein's equations then become

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T^{else}_{\mu\nu}. \tag{8}$$

This is how Einstein wrote it when he made his "greatest blunder" by putting it in to try to force his equations to give a static universe, rather than Hubble expansion.

• Cosmological constant $T_{\mu\nu} = -\rho_\Lambda g_{\mu\nu} + T^{\rm else}_{\mu\nu}$. Supernovae observations (1998) + other tests

$$\rho_\Lambda = (1.148 \pm 0.11) \times 10^{-123}$$

in natural, Planck units. $\Omega_\Lambda \approx 0.7$.

• Free-falling coordinates: can always use a coordinate transformation to take $x^\mu \to x^{\hat\mu}$ such that, at some point p, $g_{\hat\mu\hat\nu}|_p = \eta_{\hat\mu\hat\nu}$ and $\partial_{\hat\sigma}g_{\hat\mu\hat\nu}|_p = 0$, so the connection vanishes there. This shows that the metric and the connection aren't direct physical observables. But the 2nd derivative can't always be chosen to vanish at p, not if there is local curvature at p.

Counting in d spacetime dimensions. The metric has $d(d+1)/2$ components. If we expand $x^\mu = \frac{\partial x^\mu}{\partial x^{\hat\mu}}|_p x^{\hat\mu} + \ldots$ around p, the first derivatives of x^μ w.r.t. $x^{\hat\mu}$ has enough terms to set $g_{\mu\nu}|_p = \eta^{\mu\nu}$. There are $\frac{1}{2}p(p-1)$ unused components here, which are the Lorentz group transformations. At next order, $\partial^2 x^\mu/\partial x^{\hat\mu}\partial x^{\hat{n}u}$ has the correct number of components to set $\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}}|_p = 0$. At next order, $\partial_\lambda \partial_\sigma g_{\mu\nu}$ has $(\frac{1}{2}d(d+1))^2$ terms whereas $\partial^3 x^\mu / \partial x^{\hat{\mu}} \partial x^{\hat{\nu}} \partial x^{\hat{\sigma}}$ has $d^2(d+1)(d+2)/6$ terms, so the difference is $\frac{1}{12}d^2(d^2-1)$, which is 20 in $d=4$. These are the independent components of the Riemann curvature tensor.

Recall $R_{([\mu\nu][\rho\sigma])}$ is antisymmetric if one exchanges the first and second index pair. Likewise for the third and forth. Symmetric if first and second pair exchanged. Also $R_{\mu[\nu\rho\sigma]} = 0$ Can count and show number of components is $\frac{1}{12}d^2(d^2-1)$, as above.

Bianchi identity: $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$. Equivalently $\nabla_\lambda R_{\rho\sigma\mu\nu} + (\lambda \to \rho \to \sigma) + (\lambda \to \sigma \to \rho) = 0$ Contract with $g^{\nu\sigma} g^{\mu\lambda}$ to get $\nabla^\mu G_{\mu\nu} = 0$.

Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = R_{\nu\mu}$ and scalar $R = R^\lambda{}_\lambda$.

Also Weyl (or conformal) tensor, which is the Riemann tensor minus all traces:

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{d-2}(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}) + \frac{2}{(d-1)(d-2)} g_{\rho[\mu} g_{\nu]\sigma} R.$$

By construction, $C_{\rho\sigma\mu\nu}$ has the same index pair (anti) symmetry as the Riemann tensor and all traces vanish: $g^{\rho\mu} C_{\rho\sigma\mu\nu} = 0$, and likewise for any other index contraction. The Weyl tensor only exists for $d > 2$. It is invariant under conformal transformations: get same answer for $g_{\mu\nu}(x)$ and $\omega^2(x) g_{\mu\nu}(x)$.