3/22/17 Lecture 18 outline

• Recall from last time:

$$E \leftrightarrow M, \quad S \leftrightarrow A/4G, \quad T \leftrightarrow \kappa/2\pi.$$

For a Schwarzschild black hole:

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM} = 1.2 \times 10^{26} K(\frac{1g}{M}) = 6.0 \times 10^{-6} K(\frac{M_\odot}{M}).$$

Introduce proper distance η by $g_{\eta\eta} = 1$, find $\eta = \sqrt{r(r-r_H)} + r_H \cosh^{-1}(\sqrt{r/r_H}) \approx 2\sqrt{r_H(r-r_H)}$ near the horizon. Define $\omega = t/2r_H$ and the metric near r_H is

$$ds^2 \sim -\eta^2 d\omega^2 + d\eta^2 + r_H^2 d\Omega^2.$$

Rotate to Euclidean time and avoid a conical singularity at the origin by giving $i\omega$ a 2π periodicity. Use $e^{iS/\hbar} \to e^{-\beta H}$ with $\tau_E/\hbar = \beta = 1/kT$ i.e. $kT = \hbar/\tau_E$. Here it gives $kT = \hbar/8\pi GM$.

• Hawking's calculation and the closely related Unruh effect. A uniformly accelerated observer in Minkowski space sees a thermal spectrum: $T = a/2\pi$.

Consider $ds^2 = -dt^2 + dx^2$ and an observer with uniform acceleration: $\alpha t(\tau) = \sinh(\alpha\tau)$, $\alpha x(\tau) = \cosh(\alpha\tau)$, which has $\sqrt{a_\mu a^\mu} = \alpha$. Motivates a coordinate change: $at = e^{a\xi}\sinh(a\eta)$ and $ax = e^{a\xi}\cosh(a\eta)$, which covers region I with $x > |t|$. A constant acceleration path has $a\eta = \alpha\tau$ and $a\xi = \ln(a/\alpha)$ where α is the acceleration. So for $\alpha = a$, the path is $\eta = \tau$, $\xi = 0$. The metric is $ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$ Rindler space or wedge. The Killing vector ∂_η corresponds to a boost in the x direction and has norm given by $V = e^{a\xi}$ and there is a horizon at $x = t$ with surface gravity $\kappa = a$. Write Rindler space plane wave solutions in terms of $e^{ik\xi - i\omega\eta}$. The original (t, x) plane corresponds to regions I, II, III, and IV.

In Minkowski space we write $\phi = \int dk(a_k f_k + a_k^\dagger f_k^*)$ where f_k is a plane wave with positive frequency. In the Rindler coordinates we write

$$\phi = \int dk(b_k^{(I)} g_k^{(I)} + b_k^{(IV)} g_k^{(IV)} + h.c.),$$

where $g_k^{(I)} \sim e^{-i\omega\eta + ik\xi}$ is a positive frequency solution in region I, which vanishes in region IV, and $g_k^{(IV)}$ is similar in region IV vanishing in region I. We quantize as usual for the Minkowski vacuum has $a_k|0\rangle_M = 0$, whereas the Rindler vacuum has $b_k^{(I)}|0\rangle_R = b_k^{(IV)}|0\rangle_R$.

Find (Carroll):

$$b_k^{(\pm)} = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}(e^{\pi\omega/2a}c_k^{\pm} + e^{-\pi\omega/2a}c_{-k}^{(\mp)\dagger}),$$

where $\pm$ refer to the regions $\pm x > 0$ and $c_k^{(\pm)}$ are annhiliation operators in terms of left or right moving $(x - \pm t)$ modes and in the vacuum $c_k^{(\pm)}|0\rangle_M = 0$. Then ${}_M\langle 0|b_k^\dagger b_k|0\rangle_M \sim (e^{2\pi\omega/a} - 1)^{-1}$, i.e. a thermal spectrum with $T = a/2\pi$.

• Hawking's description. $\phi = \int dk(f_k a_k + h.c.)$ where f_k are positive frequency of $\mathcal{I}^-$. Also $\phi = \int dk(p_k b_k + q_k c_k + h.c.)$ where p_k are purely outgoing, giving zero on the horizon, and q_k are purely incoming, giving zero on $\mathcal{I}^+$, both with positive frequency where they are non-zero. Find $p_i = \sum_j(\alpha_{ij}f_j + \beta_{ij}\bar{f}_j)$ and $q_i = \sum_j(\gamma_{ij}f_j + \eta_{ij}\bar{f}_j$ and hence $b_i = \sum_j(\bar{\alpha}_{ij}a_j - \bar{\beta}_{ij}a_j^\dagger)$ and $c_i = \sum_j(\bar{\gamma}_{ij}a_j - \bar{\eta}_{ij}a_j^\dagger)$. Take initial vacuum $a_i|0\rangle = 0$ and then find that the outgoing mode has $\langle 0_-|b_i^\dagger b_i|0_-\rangle = \sum_j|\beta_{ij}|^2$. Consider a wave equation solution p_ω propagating backwards from $\mathcal{I}^+$ with zero Cauchy data on the event horizon. Follow solution back to $\mathcal{I}^-$. Find this is determined in terms of κ

• Related to entanglement: $\mathrm{tr}_{x<0}|0\rangle\langle 0| = Z^{-1}e^{-2\pi H}$, corresponding to $T = 1/2\pi$.

• Density matrix associated with region A, $\rho_A = \mathrm{Tr}_{\bar{A}}|\Psi\langle\rangle\Psi|$ and then $S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$. In AdS/CFT this can be computed from the Ryu Takayanagi formula, $S(A) = Area(\tilde{A})/4G$ where $\tilde{A}$ is the minimal surface in the bulk that gives A on the boundary.

• Near the horizon, take $r = r_H + \delta$, $\delta \equiv \rho^2/4r_H$, and $X = \rho\cosh(t/2r_H)$ and $T = \rho\sinh(t/2r_H)$. Let $U = T - X$ and $V = T + X$. Region 1 is $U < 0$, $V > 0$, region 2 is $U > 0$, $V > 0$ etc.

Infalling observer has $d\tau \propto e^{-t/r_H}dt$. Infalling observer has τ frequencies ν and outside observer has t frequency modes ω. Consider 2d KG field and let $u, v = t \mp r_*$ with $r_* = r + r_H \ln(r - r_H)$. Expand $\phi = \int(a_\nu e^{-i\nu U} + h.c.)$ or $\phi = \int(b_\omega e^{-i\omega u} + h.c.)$. Then

$$b_\omega = \int d\nu(\alpha_{\omega\nu}a_\nu + \beta_{\omega\nu}a_\nu^\dagger).$$

where $\alpha_{\omega\nu} = 2r_H(\omega/\nu)^{1/2}(2r_H\nu)^{2ir_H\omega}e^{-\pi r_H\omega}\Gamma(-2ir_H\omega)$ and $\beta_{\omega\nu} = e^{-2\pi r_H\omega}\alpha_{\omega\nu}$. Then $|0\rangle$ with $a_\nu|0\rangle = 0$ has

$$\langle 0|b_\omega^\dagger b_{\omega'}|0\rangle = \frac{2\pi\delta(\omega - \omega')}{e^{\omega/T_H} - 1}.$$

Let b_ω and $b^\dagger$ be the operators for modes in region I and $\tilde{b}_\omega$ and $\tilde{b}_\omega^\dagger$ those for region II. The final state is $|0\rangle_a \sim \exp(\int_0^\infty d\omega/2\pi e^{-\omega/2T_H}b_\omega^\dagger\tilde{b}_\omega^\dagger)|0\rangle_b$.

• The typical Hawking quanta has energy $T_H \sim 1/r_H \sim 1/GM$ so if the entire BH evaporates the number of quanta will be $\sim M/(1/GM) \sim GM^2$. So the entropy of the Hawking quanta goes from $0 \to GM^2$ and that of the BH goes from $GM^2 \to 0$.