3/21/17 Lecture 17 outline

• Recall from last time: Kerr black holes:

$$ds^2 = -\Sigma^{-1}(\Delta - a^2\sin^2\theta)dt^2 - 2a\Sigma^{-1}\sin^2\theta(r^2+a^2-\Delta)dtd\phi +$$

$$+\Sigma^{-1}((r^2+a^2)^2 - \Delta a^2\sin^2\theta)\sin^2\theta d\phi^2 + \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2.$$

Here $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 + Q^2 - 2GMr$ and the gauge field is

$$A_\mu dx^\mu = -Qr\Sigma^{-1}(dt - a\sin^2 d\phi),$$

where Q is the electric charge, as measured by the flux through a sphere at infinity, and $Ma = J$ is the angular momentum (as measured through a large sphere at infinity). The metric is t and ϕ independent, so it admits Killing vectors $K^\mu = \partial_t^\mu$ and $R^\mu = \partial_\phi^\mu$. The $dtd\phi$ cross term means that it is stationary but not static, corresponding to the BHs rotation, which frame-drags spacetime along with it.

• Compare the conformal diagrams of eternal Schwarzschild vs eternal Kerr Newmann.

• Continue from last time. Consider a massive particle on a geodesic in the Kerr geometry with $p^\mu = mu^\mu$. The conserved quantities associated with the Killing vectors are

$$E = -K_\mu p^\mu = m(1 - \frac{2GMr}{\Sigma})\frac{dt}{d\tau} + \frac{2GmMar}{\Sigma}\frac{d\phi}{d\tau},$$

$$L = R_\mu p^\mu = -\frac{2GMmar}{\Sigma}\sin^2\theta\frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\frac{d\phi}{d\tau}.$$

• Extracting energy from a Kerr black hole. In a free falling frame, energy and momentum conservation is $p^\mu_{in} = p^\mu_{out} + p^\mu_{BH}$. Use K^μ to get energies: $E_{out} = E_{in} - E_{BH}$. But if the particle going into the BH is inside the ergosphere, then $K_\mu K^\mu = g_{00} > 0$ and $E_{BH} < 0$. The outgoing particle can have more energy than the incoming one – it has extracted energy from the ergosphere. Consider an observer inside the ergosphere with $u^\mu_{obs} = u^t_{obs}(K^\mu + \Omega_{obs}R^\mu)$. They must measure a positive energy going into the BH, so $-(K + \Omega_{obs}R)\cdot p_{BH} \geq 0$. This gives $E_{BH} \geq \Omega_{obs}L_{BH}$ where $L_{BH} = m_{BH}\ell_{BH}$ is the angular momentum of the particle that fell into the black hole. Since $\Omega_{obs} > 0$, negative E_{BH} requires negative L_{BH}, so the energy extraction also extracts angular momentum from the BH. This is called the Penrose process. The black hole has $\delta M = E_{BH}$ and $\delta J = L_{BH}$. We will see that the area of the black hole increases in the Penrose process, even though energy and angular momentum are being extracted. This is a special case of the general black-hole area increase theorems of classical GR. This is the starting point for black hole thermodynamics: black holes have an entropy $S = A/4G$, and the area-increase theorem is then the 2nd law of thermodynamics. This was a starting point for Hawking's observation that black holes are quantumly hot, and radiate like a thermal blackbody with a temperature T_H.

• Field theory analog of the Penrose process: super-radiant scattering. Consider scattering a wave of the form $\phi = \Re(\phi_0(r,\theta)e^{-i\omega t}e^{im\phi})$. If $0 < \omega < m\Omega_H$, the transmitted wave carries negative energy into the black hole, and the reflected wave thus has a larger amplitude than the incoming wave. The energy current $J^\mu = T_{\mu\nu}K^\nu$ has average flux through the horizon given by $\langle T_{\mu\nu}\chi^\mu K^\nu\rangle = \frac{1}{2}\omega(\omega - m\Omega_H)|\phi_0|^2$.

• The area of the outer horizon, at $r_+ = GM\sqrt{G^2M^2 - a^2}$ is computed from the induced metric, setting $dr = dt = 0$. The result is $A = \int \sqrt{|\gamma|}d\theta d\phi = 4\pi(r_+^2 + a^2)$. Define (with $J = Ma$)

$$M_{irr}^2 \equiv \frac{A}{16\pi G^2} = \tfrac{1}{2}(M^2 + \sqrt{M^4 - (J/G)^2}).$$

The change is

$$\delta M_{irr} = \frac{a}{4GM_{irr}\sqrt{G^2M^2 - a^2}}(\Omega_H^{-1}\delta M - \delta J),$$

and $L^{(2)} < E^{(2)}/\Omega_H$ implies $\delta J < \delta M/\Omega_H$, and hence $\delta M_{irr} > 0$.

Write it as

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J, \leftrightarrow dE = TdS - pdV$$

where κ is the surface gravity. Recall it is obtained from the Killing vector $\chi = K^\mu + \Omega_H R^\mu$ of the horizon

$$\kappa = \sqrt{-\tfrac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)} = \frac{\sqrt{G^2M^2 - a^2}}{2GM(GM + \sqrt{G^2M^2 - a^2})}.$$

Black hole thermodynamics has

$$E \leftrightarrow M, \quad S \leftrightarrow A/4G, \quad T \leftrightarrow \kappa/2\pi.$$

Recall that, at the horizon for a stationary configuration, a Killing vector χ^μ is null, and $\chi^\mu\nabla_\mu\chi^\nu = -\kappa\chi^\nu$ where κ is constant on the horizon. Recall that for a static observer (consider a non-rotating BH for the moment) $K^\mu = VU^\mu$ with $V = \sqrt{-K_\mu K^\mu}$ and photons e.g. have $\omega_2/\omega_1 = V_1/V_2$ and the four-acceleration is $a_\mu = U^\sigma\nabla_\sigma U^\mu = \nabla_\mu \ln V$, with magnitude $a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V}$ and then $\kappa = Va$ is the acceleration at the end of a string at infinity.

- For a Schwarzschild black hole:

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM} = 1.2 \times 10^{26} K(\frac{1g}{M}) = 6.0 \times 10^{-6} K(\frac{M_\odot}{M}).$$

Introduce proper distance η by $g_{\eta\eta} = 1$, find $\eta = \sqrt{r(r - r_H)} + r_H \cosh^{-1}(\sqrt{r/r_H}) \approx 2\sqrt{r_H(r - r_H)}$ near the horizon. Define $\omega = t/2r_H$ and the metric near r_H is

$$ds^2 \sim -\eta^2 d\omega^2 + d\eta^2 + r_H^2 d\Omega^2.$$

Rotate to Euclidean time and avoid a conical singularity at the origin by giving $i\omega$ a 2π periodicity. Use $e^{iS/\hbar} \rightarrow e^{-\beta H}$ with $\tau_E/\hbar = \beta = 1/kT$ i.e. $kT = \hbar/\tau_E$. Here it gives $kT = \hbar/8\pi GM$.