3/15/17 Lecture 16 outline

- Recall from last time: Kerr black holes:

$$ds^2 = -\Sigma^{-1}(\Delta - a^2\sin^2\theta)dt^2 - 2a\Sigma^{-1}\sin^2\theta(r^2+a^2-\Delta)dtd\phi +$$

$$+\Sigma^{-1}((r^2+a^2)^2 - \Delta a^2\sin^2\theta)\sin^2\theta d\phi^2 + \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2.$$

Here $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 + Q^2 - 2GMr$ and the gauge field is

$$A_\mu dx^\mu = -Qr\Sigma^{-1}(dt - a\sin^2 d\phi),$$

where Q is the electric charge, as measured by the flux through a sphere at infinity, and $Ma = J$ is the angular momentum (as measured through a large sphere at infinity). The metric is t and ϕ independent, so it admits Killing vectors $K^\mu = \partial_t^\mu$ and $R^\mu = \partial_\phi^\mu$. The $dtd\phi$ cross term means that it is stationary but not static, corresponding to the BHs rotation, which frame-drags spacetime along with it.

For $r \gg M$ and $r \gg a$, note that

$$ds^2 \approx (1 - \frac{2GM}{r})dt^2 + (1 + \frac{2GM}{r})dr^2 + r^2 d\Omega^2 - \frac{4Ma}{d^2}\sin^2\theta(rd\phi)dt + \ldots.$$

Recall for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small, that $h_{00} = -2\Phi$, $h_{0i} \equiv w_i$ etc and $H^i \equiv \epsilon^{ijk}\partial_h w_k$ is analogous to a magnetic field in that it leads to a $\dot{\vec{p}} = E\vec{v} \times \vec{H} + \ldots$ term, which here is a rotational term $\dot{\vec{p}} = \vec{\Omega} \times \vec{p} + \ldots$ with $\vec{\Omega}$ pointing in the $\widehat{\phi}$, i.e. $\widehat{z}$ direction.

The full Kerr metric exhibits several interesting locations:

(i) The place where $g_{00} = 0$ is called the stationary limit surface.

(ii) The places where $g^{rr} = 0$ are event horizons.

(iii) The places where $\Sigma = 0$ are singularities for $M, a \neq 0$.

As a warmup consider first $Q = M = 0$. Then the Kerr solution is simply Minkowski space in ellipsoidal coordinates: $x = \sqrt{r^2+a^2}\sin\theta\cos\phi$, $y = \sqrt{r^2+a^2}\sin\theta\sin\phi$, $z = r\cos\theta$. Then $r = 0$ is a two dimensional disk of radius a and its intersection with $\theta = \pi/2$ is the ring at the boundary of this disk.

Now consider $M \neq 0$ and $a \neq 0$. Computing $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ find that it is singular at $\Sigma = 0$, i.e. at $r = 0$, $\theta = \pi/2$, i.e. at the above-mentioned ring of radius a in the $z = 0$ plane (orthogonal to the angular momentum). So the spinning spreads out the point singularity of Schwarzschild into a ring singularity. The event horizons are at $\Delta = 0$. The stationary limit surface is at $\Delta = a^2\sin^2\theta$, so it is outside of the event horizon and they touch at the N and S poles. The region between the event horizon and the stationary limit surface is the ergosphere.

If $Q^2 + a^2 > M^2$, there is no $\Delta = 0$ solution, hence a naked singularity. Can go backwards in time and have closed timeline curves in that case by circling around the singularity. According to the cosmic censorship conjecture, such black holes never form from smooth physical configurations. So consider $Q^2 + a^2 \leq M^2$, which has horizons at $\Delta = 0$ i.e. $r = r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, the outer and inner horizon. These are coordinate singularities and the space-time can be extended past them. Spacetime can be extended to negative r. For the negative r region, there are closed time-like curves at the ring singularity, e.g. wind in ϕ: $ds^2 \approx a^2(1 + 2GM/r)d\phi^2$ which can be negative for small negative r. Since $\Delta \neq 0$ for $r < 0$, there are no horizons and naked singularities in that asymptotic region.

Because Kerr is stationary but not static, the event horizons at $r_\pm$ are not Killing horizons for the asymptotic time-translation Killing vector $K = \partial_t$. The norm of K^μ is $K_\mu K^\mu = -\Sigma^{-1}(\Delta - a^2 \sin^2\theta)$, so at the outer horizon $K_\mu K^\mu = a^2 \Sigma^{-1} \sin^2\theta \geq 0$: it is space like at the outer horizon, and null at the poles. The stationary limit surface is where $K_\mu K^\mu = 0$, i.e. at $(r_{s.l.s} - GM)^2 = G^2M^2 - a^2\cos^2\theta$, which has $r_{s.l.s.} \geq r_+$, touching the outer horizon at the north and south poles. The region between $r_{s.l.s.}$ and r_+ is the ergosphere. Once inside the ergosphere, it is impossible to not rotate with the BH in the ϕ direction, but you can still move either to or away from the event horizon.

The null vector at $r = r_+$ is $\ell^\mu = K^\mu + \Omega_H R^\mu$, with $\Omega_H = a/(r_+^2 + a^2)$ interpreted as the angular velocity of the horizon itself. The null ℓ^μ are tangent vectors to the light rays that form the horizon. These light rays are rotating with angular velocity Ω_H; this is frame dragging.

- Consider a photon emitted in the ϕ direction at $\theta = \pi/2$ has $ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2$, so

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{(\frac{g_{t\phi}}{g_{\phi\phi}})^2 - \frac{g_{tt}}{g_{\phi\phi}}}.$$

At the stationary limit surface the two solutions are $\frac{d\phi}{dt} = 0$ and $\frac{d\phi}{dt} = a/(2G^2M^2 + a^2)$, corresponding to going against the rotation or with the rotation. The angular velocity of the event horizon is $\Omega_H = (\frac{d\phi}{dt})_-(r_+) = a/(r_+^2 + a^2)$.

Consider an observer who, with help from a rocket, tries to keep their r, θ, ϕ values unchanging. In Schwarzschild, this can be done for $r > 2GM$. Now consider the case for Kerr, trying to keep $u^\mu_{obs} = (u^t_{obs}, 0, 0, 0)$ with $u_\mu u^\mu = -g_{00}{u^t_{obs}}^2 = -1$. The place where

$g_{00} = 0$ defines the stationary limit surface, r_{sls}. For $r < r_{sls}$ it is impossible to have u_{obs} with only time-like components, even with an arbitrarily powerful rocket. Inside this region is the ergosphere, where $u_{obs} = u^t_{obs}(1, 0, 0, \Omega_{obs})$, rotating in the ϕ direction along with the BH.

• Draw pictures of light fronts at different places in the Kerr geometry.

• Consider a geodesic orbit in the Kerr geometry, say at $\theta = \pi/2$. For simplicity, take $q_{obj} = Q_{BH} = 0$. Thanks to the t and ϕ translation symmetry, there are conserved quantities $e = -K \cdot u$ and $\ell = R \cdot u$, the energy and angular momentum per unit mass. It turns out that there is another conserved quantity: a Killing tensor

$$K_{\mu\nu} = 2\Sigma\ell_{(\mu}n_{\nu)} + r^2 g_{\mu\nu}$$

where $\ell^\mu = (r^2 + a^2)\Delta^{-1}\partial_t^\mu + a\Delta^{-1}\partial_\phi^\mu + \partial_r^\mu$ and $n_\mu = \frac{1}{2}(r^2 + a^2)\Sigma^{-1}\partial_t^\mu + \frac{1}{2}a\Sigma^{-1}\partial_\phi^\mu - \frac{1}{2}\Delta\Sigma^{-1}\partial_r^\mu$ so $C = K_{\mu\nu}u^\mu u^\nu$ is also a constant of the motion.

Take $g_{\mu\nu}u^\mu u^\nu \equiv -\kappa$ with $\kappa = 1$ for massive orbiters and $\kappa = 0$ for massless. Then

$$\tfrac{1}{2}(e^2 - 1) = \tfrac{1}{2}(\frac{dr}{d\tau})^2 + V_{eff},$$

$$V_{eff} = -\kappa\frac{GM}{r} + \frac{\ell^2 - a^2(e^2 - \kappa)}{2r^2} - \frac{M(\ell - ea)^2}{r^3}.$$

Note that it is not $\ell \to -\ell$ symmetric: the effective potential differs whether the orbiter's rotation is aligned or anti-aligned with that of the BH. The sign of the potential helps to avoid violating cosmic censorship, i.e. avoid having $a > M$, because particles with ℓ too big can't fall in.