3/13/17 Lecture 15 outline

Recall the definition of the future and past event horizons of a space-time: null hyper surfaces intersecting $\mathcal{I}^{\pm}$.

A stationary metric has a killing vector $K^\mu = \partial_t^\mu$. We can choose metric s.t. $\partial_t g_{\mu\nu} = 0$. Choose r such that $r = const$ hyper surfaces remain timeline until some r_H, where they become null. This is an event horizon. The normal to the hupersurface is $\partial_\mu r$ and its norm is $g^{\mu\nu}\partial_\mu r \partial_\nu r = g^{rr}$, so the event horizon has $g^{rr}(r_H) = 0$.

Recall for asymptotically flat space-time, normalize K_μ such that $K_\mu K^\mu(r\to\infty) \to -1$. For a static observer, $K^\mu = V(x)U^\mu$, where $V = -\sqrt{K_\mu K^\mu}$. Recall the energy of a photon is $E = -p_\mu K^\mu$ and the frequency measured by an observer with U^μ is $\omega = -p_\mu U^\mu$, so $\omega = E/V$. A static observer hovering at fixed spatial coordinates has acceleration $a^\mu = U^\sigma \nabla_\sigma U^\mu = \nabla^\mu \ln V$. The acceleration magnitude is $a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V}$. The surface gravity at the event horizon is $\kappa = Va = \sqrt{\nabla_\mu V \nabla^\mu V}$. Picture a string from a static object at the horizon connecting to an observer at infinity, then the surface gravity is the acceleration of the end at infinity.

Define the stationary limit surface to be where $K_\mu K^\mu$ changes sign, i.e. where $g_{00} = 0$. Beyond there, one cannot remain at fixed spatial coordinates. For a Schwarzschild black hole both the stationary limit surface and the event horizon are at $r = 2GM$.

- Charged Reissner-Nordstrom black holes:

$$ds^2 = -H(r)dt^2 + H(r)^{-1}dr^2 + r^2 d\Omega^2,$$

$$H = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2},$$

where Q is the electric charge and P is the magnetic charge, and $E_r = Q/r^2$ and $B_r = P/r^2$. The event horizon is where $H = 0$ and there are two solutions:

$$r_\pm = GM \pm \sqrt{G^2M^2 - G(Q^2+P^2)}.$$

So there are two, one, or zero solutions depending on if GM^2 is bigger than, equal, or less than $Q^2 + P^2$.

The conformal diagram of the $Q^2+P^2 > GM^2$ case looks similar to Minkowski space, but with a naked singularity at $r = 0$. According to the cosmic censorship conjecture, such black holes can never form.

Now consider the $GM^2 > Q^2 + P^2$ case. The metric has coordinate singularities at $r_\pm$, which can be removed by an analog of Kruskal coordinates. $r_\pm$ are both null event horizons. The singularity at $r = 0$ is different from Schwarzschild: it is time-like instead of space-like. If you fall in, the inward r direction switches from pointing in a space like direction to being time-like at r_+, just as in Schwarzschild, but then it switches back to being space-like after crossing r_-. So once you cross r_- you can avoid hitting $r = 0$. You could even cross back to $r > r_-$. Then your time direction would point towards *increasing* r, and you'll be spit out at r_+, like a white hole. Draw conformal diagram.

Finally, consider the extremal case $GM = Q^2 + P^2$. So one event horizon at $r = GM$, but r coordinate is never time-like: it is null at $r = GM$ but space-like on either side. $r = 0$ is again a time-like line. Draw the conformal diagram. The extremal case has an exact cancellation between the gravitational attraction and the electric replusion. Correspondingly, it is easy to construct exact solutions of GR with arbitrary numbers of extremal BHs (something that cannot be done with other solutions e.g. Schwarzschild). These BHs can be regarded as the bosonic part of supersymmetric solutions called BPS configurations, which have a generalization of the no force property.

- Kerr black holes:

$$ds^2 = -\Sigma^{-1}(\Delta - a^2 \sin^2\theta)dt^2 - 2a\Sigma^{-1}\sin^2\theta(r^2 + a^2 - \Delta)dtd\phi +$$

$$+\Sigma^{-1}((r^2 + a^2)^2 - \Delta a^2 \sin^2\theta)\sin^2\theta d\phi^2 + \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2.$$

Here $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 + Q^2 - 2GMr$ and the gauge field is

$$A_\mu dx^\mu = -Qr\Sigma^{-1}(dt - a\sin^2 d\phi),$$

where Q is the electric charge, as measured by the flux through a sphere at infinity, and $Ma = J$ is the angular momentum (as measured through a large sphere at infinity). The metric is t and ϕ independent, so it admits Killing vectors $K^\mu = \partial_t^\mu$ and $R^\mu = \partial_\phi^\mu$. The $dtd\phi$ cross term means that it is stationary but not static, corresponding to the BHs rotation, which frame-drags spacetime along with it.

For $r \gg M$ and $r \gg a$, note that

$$ds^2 \approx (1 - \frac{2GM}{r})dt^2 + (1 + \frac{2GM}{r})dr^2 + r^2 d\Omega^2 - \frac{4Ma}{d^2}\sin^2\theta(rd\phi)dt + \dots .$$

Recall for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small, that $h_{00} = -2\Phi$, $h_{0i} \equiv w_i$ etc and $H^i \equiv \epsilon^{ijk}\partial_h w_k$ is analogous to a magnetic field in that it leads to a $\dot{\vec{p}} = E\vec{v} \times \vec{H} + \ldots$ term, which here is a rotational term $\dot{\vec{p}} = \vec{\Omega} \times \vec{p} + \ldots$ with $\vec{\Omega}$ pointing in the $\widehat{\phi}$, i.e. $\widehat{z}$ direction.

The full Kerr metric exhibits several interesting locations:

(i) The place where $g_{00} = 0$ is called the stationary limit surface.

(ii) The places where $g^{rr} = 0$ are event horizons.

(iii) The places where $\Sigma = 0$ are singularities for $M, a \neq 0$.

As a warmup consider first $Q = M = 0$. Then the Kerr solution is simply Minkowski space in ellipsoidal coordinates: $x = \sqrt{r^2 + a^2} \sin\theta\cos\phi$, $y = \sqrt{r^2 + a^2}\sin\theta\sin\phi$, $z = r\cos\theta$. Then $r = 0$ is a two dimensional disk of radius a and its intersection with $\theta = \pi/2$ is the ring at the boundary of this disk.

Now consider $M \neq 0$ and $a \neq 0$. Computing $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ find that it is singular at $\Sigma = 0$, i.e. at $r = 0$, $\theta = \pi/2$, i.e. at the above-mentioned ring of radius a in the $z = 0$ plane (orthogonal to the angular momentum). So the spinning changes the point singularity to a ring singularity. The horizon is at $\Delta = 0$. If $Q^2 + a^2 > M^2$, there is no $\Delta = 0$ solution, hence a naked singularity. Can go backwards in time and have closed timeline curves in that case by circling around the singularity. For $Q^2 + a^2 \leq M^2$, $\Delta = 0$ for $r = r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, the outer and inner horizon. These are coordinate singularities and the space-time can be extended past them. Spacetime can be extended to negative r. Closed time-like curves at the ring singularity, e.g. wind in ϕ: $ds^2 \approx a^2(1 + 2GM/r)d\phi^2$ which can be negative for small negative r.

Null vector at $r = r_+$ is $\ell^\mu = K^\mu + \Omega_H R^\mu$, with $\Omega_H = a/2Mr_+$. The null ℓ^μ are tangent vectors to the light rays that form the horizon. These light rays are rotating with angular velocity Ω_H; this is frame dragging. A photon emitted in the ϕ direction at $\theta = \pi/2$ has $ds^2 = 0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2$, so

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{(\frac{g_{t\phi}}{g_{\phi\phi}})^2 - \frac{g_{tt}}{g_{\phi\phi}}}.$$

At the stationary limit surface the two solutions are $\frac{d\phi}{dt} = 0$ and $\frac{d\phi}{dt} = a/(2G^2M^2 + a^2)$, corresponding to going against the rotation or with the rotation. The angular velocity of the event horizon is $\Omega_H = (\frac{d\phi}{dt})_-(r_+) = a/(r_+^2 + a^2)$.

Consider an observer who, with help from a rocket, tries to keep their r, θ, ϕ values unchanging. In Schwarzschild, this can be done for $r > 2GM$. Now consider the case for Kerr, trying to keep $u^\mu_{obs} = (u^t_{obs}, 0, 0, 0)$ with $u_\mu u^\mu = -g_{00}{u^t_{obs}}^2 = -1$. The place where

$g_{00} = 0$ defines the stationary limit surface, r_{sls}. For $r < r_{sls}$ it is impossible to have u_{obs} with only time-like components, even with an arbitrarily powerful rocket. Inside this region is the ergosphere, where $u_{obs} = u^t_{obs}(1, 0, 0, \Omega_{obs})$, rotating in the ϕ direction along with the BH.

Consider a geodesic orbit in the Kerr geometry, say at $\theta = \pi/2$. There is conserved $e = -K \cdot u$ and $\ell = R \cdot u$, where $u \cdot u = -1$ for a massive, time-like orbiter. Find

$$\tfrac{1}{2}(e^2 - 1) = \tfrac{1}{2}(\frac{dr}{d\tau})^2 + V_{eff},$$

$$V_{eff} = -\frac{GM}{r} + \frac{\ell^2 - a^2(e^2-1)}{2r^2} - \frac{M(\ell - ea)^2}{r^3}.$$

Note that it is not $\ell \to -\ell$ symmetric: the effective potential differs whether the orbiter's rotation is aligned or anti-aligned with that of the BH. The sign of the potential helps to avoid violating cosmic censorship, i.e. avoid having $a > M$, because particles with ℓ too big can't fall in.

Extracting energy from a Kerr black hole. In a free falling frame, energy and momentum conservation is $p^\mu_{in} = p^\mu_{out} + p^\mu_{BH}$. Use K^μ to get energies: $E_{out} = E_{in} - E_{BH}$. But if the particle going into the BH is inside the ergosphere, then $K_\mu K^\mu = g_{00} > 0$ and $E_{BH} < 0$. The outgoing particle can have more energy than the incoming one – it has extracted energy from the ergosphere. Consider an observer inside the ergosphere with $u^\mu_{obs} = u^t_{obs}(K^\mu + \Omega_{obs} R^\mu)$. They must measure a positive energy going into the BH, so $-(K + \Omega_{obs} R) \cdot p_{BH} \geq 0$. This gives $E_{BH} \geq \Omega_{obs} L_{BH}$ where $L_{BH} = m_{BH} \ell_{BH}$ is the angular momentum of the particle that fell into the black hole. Since $\Omega_{obs} > 0$, negative E_{BH} requires negative L_{BH}, so the energy extraction also extracts angular momentum from the BH. This is called the Penrose process. We will see that the area of the black hole increases in the Penrose process, even though energy and angular momentum are being extracted. This is a special case of the general black-hole area increase theorems of classical GR. This is the starting point for black hole thermodynamics: black holes have an entropy $S = A/4G$, and the area-increase theorem is then the 2nd law of thermodynamics. This was a starting point for Hawking's observation that black holes are quantumly hot, and radiate like a thermal blackbody with a temperature T_H. More on this in a later week.

Because Kerr is stationary but not static, the event horizons at $r_\pm$ are not Killing horizons for the asymptotic time-translation Killing vector $K = \partial_t$. The norm of K^μ is $K_\mu K^\mu = -\Sigma^{-1}(\Delta - a^2 \sin^2\theta)$, so at the outer horizon $K_\mu K^\mu = a^2 \Sigma^{-1} \sin^2\theta \geq 0$: it is space like at the outer horizon, and null at the poles. The stationary limit surface is where

$K_\mu K^\mu = 0$, i.e. at $(r_{s.l.s} - GM)^2 = G^2M^2 - a^2\cos^2\theta$, which has $r_{s.l.s.} \geq r_+$, touching the outer horizon at the north and south poles. The region between $r_{s.l.s.}$ and r_+ is the ergosphere. Once inside the ergosphere, it is impossible to not rotate with the BH in the ϕ direction, but you can still move either to or away from the event horizon.

If you go inside the ring singularity, you exit to another asymptotically flat space-time, but not an identical copy of the original one. The new space-time is like Kerr with $r < 0$ so $\Delta \neq 0$ and there are no horizons.

Compare the conformal diagrams of eternal Schwarzschild vs eternal Kerr Newmann.