3/1/17 Lecture 14 outline

• Recall from last time: **Trapped surface:** a compact, two-dimensional, space like surface such that θ on both sets of geodesics (ingoing and outgoing) orthogonal to T are everywhere negative. The various singularity theorems connect trapped surfaces to singularities. E.g. in the Schwarzschild black hole in region II all surfaces are trapped.

• Black hole no hair (classically) theorem: stationary, asymptotically flat solutions of GR coupled to E&M that are nonsingular outside of the event horizon are fully specified by gauge charges (not global symmetry charges). The Schwarzschild black hole is specified by M, which is related to the energy. Other gauge charges are electric charge and angular momentum. The Reissner-Nordstrom (1918) black hole generalizes Schwarzschild to include electric charge. The Kerr black hole (1963) includes instead angular momentum. The Newmann et al solution (1965) includes both mass, electric charge, and angular momentum.

• Hawking's area theorem: assuming the weak energy condition ($T_{\mu\nu}t^\mu t^\nu \geq 0$ for all time-like t^μ) and cosmic censorship, the area of a future event horizon in an asymptotically flat space-time is non-decreasing. Implies e.g. that black holes cannot bifurcate.

• Event horizon, as pictured in a Penrose diagram: a null hyper surface beyond which timelike curves cannot escape to infinity.

Every event horizon Σ in a stationary, asymptotically flat space-time is a Killing horizon for some Killing vector field, i.e. the Killing vector becomes null there. If the space-time is static, this Killing vector field is simply $K^\mu = \partial_t^\mu$. If the space-time is stationary but not static the event horizon is the Killing horizon for $K^\mu + \Omega_H R^\mu$ where R^μ is the rotational Killing field $R_\mu = \partial_\phi^\mu$ and Ω_H is some constant.

For asymptotically flat space-time, normalize K_μ such that $K_\mu K^\mu(r \to \infty) \to -1$. For a static observer, $K^\mu = V(x)U^\mu$, where $V = -\sqrt{K_\mu K^\mu}$. Recall the energy of a photon is $E = -p_\mu K^\mu$ and the frequency measured by an observer with U^μ is $\omega = -p_\mu U^\mu$, so $\omega = E/V$.

Consider an observer who, with help from a rocket, tries to keep their spatial coordinates unchanging. In Schwarzschild, this can be done for $r > 2GM$. Now consider the case for Kerr, trying to keep $u^\mu_{obs} = (u^t_{obs}, 0, 0, 0)$ with $u_\mu u^\mu = -g_{00}{u^t_{obs}}^2 = -1$. The place where $g_{00} = 0$ defines the stationary limit surface, r_{sls}. For $r < r_{sls}$ it is impossible to have u_{obs} with only time-like components, even with an arbitrarily powerful rocket. Inside this region is the ergosphere, where $u_{obs} = u^t_{obs}(1,0,0,\Omega_{obs})$, rotating in the ϕ direction along with the BH.

A static observer hovering at fixed spatial coordinates has acceleration $a^\mu = U^\sigma \nabla_\sigma U^\mu = \nabla^\mu \ln V$. The acceleration magnitude is $a = \sqrt{a_\mu a^\mu} = V^{-1}\sqrt{\nabla_\mu V \nabla^\mu V}$. The surface gravity at the event horizon is $\kappa = Va = \sqrt{\nabla_\mu V \nabla^\mu V}$. Picture a string from a static object at the horizon connecting to an observer at infinity, then the surface gravity is the acceleration of the end at infinity.

E.g. for Schwarzschild, $K^\mu = (1,0,0,0)$, $U^\mu = ((1-2GM/r)^{-1/2},0,0,0)$, $V = \sqrt{1-2GM/r}$, and then $\kappa = 1/4GM$. Bigger black hole has smaller surface gravity at the horizon.

- Mass charge and spin. In E&M, $\nabla_\nu F^{\mu\nu} = J^\mu_e$, the conserved electric charge is

$$Q = -\int_\Sigma d^3x \sqrt{\gamma} n_\mu J^\mu_e,$$

where γ_{ij} is the induced metric on some space like surface Σ and n^μ is a future-pointing unit vector, and the minus sign is needed with these conventions. By Maxwell s equations and Gauss' law,

$$Q = -\int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} n_\mu \sigma_\nu F^{\mu\nu}.$$

Can likewise define magnetic charge P (if magnetic monopoles exist) by replacing $F^{\mu\nu} \to \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. In terms of forms, writing $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$, can write

$$Q = \int_{\partial\Sigma} *F, \qquad P = \int_{\partial\Sigma} F.$$

Now try to analogously define the energy. We already discussed a bit about how energy is subtle in GR, because it is a gauge instead of a global charge. The first attempt is via $J^\mu_T = K_\nu T^{\mu\nu}$, where K^μ is a time-like Killing vector field. This satisfies $\nabla_\mu J^\mu_T = 0$, so can get a conserved energy via

$$E_T = \int_\Sigma d^3x \sqrt{\gamma} n_\mu J^\mu_T.$$

But this is no good, e.g. $T_{\mu\nu} = 0$ for a Schwarzschild black hole (except at the origin) so this would give $E_T = 0$. A better choice is $J^\mu_R = K_\nu R^{\mu\nu} = 8\pi G K_\nu (T^{\mu\nu} - \frac{1}{2}Tg^{\mu\nu})$. By the Bianchi identity, $\nabla_\mu R^{\mu\nu} = \frac{1}{2}\nabla^\nu R$ and then using the Killing vector equation $\nabla_{(\mu}K_{\nu)} = 0$ it follows that J^μ_R is conserved, $\nabla_\mu J^\mu_R = \frac{1}{2}K_\nu \nabla^\nu R = 0$, where the last identity follows from a property of Killing vectors that you can prove as an exercise (Carroll):

$\nabla_\mu \nabla_\sigma K^\rho = R^\rho_{\sigma\mu\nu} K^\nu$ which can be used to show that R does not change along a Killing vector field. Using these properties of Killing vectors, show $J^\mu_R = \nabla_\nu(\nabla^\mu K^\nu)$. This leads to the Komar integral:

$$E_R = \frac{1}{4\pi G}\int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu_R = \frac{1}{4\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu K^\nu.$$

E.g. for Schwarzschild, have $n_0 = -(1-2GM/r)^{1/2}$ and $\sigma_r = (1-2GM/r)^{-1/2}$ and then $K^\mu = (1,0,0,0)$ has $n_\mu\sigma_\nu\nabla^\mu K^\nu = -\nabla^0 K^r = -g^{00}\Gamma^r_{00}K^0 = GM/r^2$, giving $E_R = (4\pi G)^{-1}\int d^2\Omega r^2(GM/r^2) = M$. There is another definition of energy,

$$E_{ADM} = \frac{1}{16\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}\sigma^i(\partial_j h^j_i - \partial_i h^j_j),$$

when the asymptotic metric is of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ a small perturbation. If $h_{\mu\nu}$ is time independent, then the Komar integral and the ADM energy agree.

Now define angular momentum for a system with rotational symmetry around say the $\hat{z}$ axis, i.e. a Killing vector $R = \partial\phi$. In analogy with the above, take $J^\mu_\phi = R_\nu R^{\mu\nu}$ and

$$J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\gamma^{(2)}}n_\mu\sigma_\nu\nabla^\mu R^\nu,$$

where $R^\nu = (\partial_\phi)^\nu$ is the Killing vector.