2/27/17 Lecture 13 outline

• Recall from last time: the Schwarzschild solution in Eddington Finkelstein coordinates, $t = v - r - 2GM \log|\frac{r}{2GM} - 1|$. Then the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Schwarzschild time t ends at the horizon, but the coordinate v keeps on going, just like the proper time of an infalling observer. (Proper time is coordinate independent.) The horizon, $r = 2GM$, $v =$ constant, is a null surface.

Kruskal coordinates: $X^2 - T^2 > -1$ and

$$(-1 + r/2GM)e^{r/2GM} = X^2 - T^2, \qquad t/2GM = \ln(\frac{T+X}{X-T}).$$

$$ds^2 = 32\frac{M^3 e^{r/2GM}}{r}(-dT^2 + dX^2) + r^2 d^2\Omega_2.$$

Regions I,II, III, IV in (T, X) plane.

• Now discuss non-eternal black holes. Then regions III and IV are not present.

Consider dust (pressureless matter) falling in Schwarzschild geometry: the geodesic equation gives $r(\tau) = (3/2)^{2/3}(2GM)^{1/3}(\tau_* - \tau)^{2/3}$. Continue via $v(r)$ to $r < 2GM$ and find, once horizon is passed, matter hits $r = 0$ in proper time $\delta\tau = 4GM/3$.

Letting $r = R(t)$ on the surface of the star,

$$ds^2 = \left[(1 - \frac{2GM}{R}) - (1 - \frac{2GM}{R})\dot{R}^2\right] dt^2 + R^2 d\Omega^2, \qquad \dot{R} \equiv dR/dt.$$

Spherical symmetry implies $d\Omega^2 = 0$ for a point on the surface, and zero pressure implies $ds^2 = -d\tau^2$. Conservation of energy gives $e = (1 - \frac{2GM}{R})\frac{dt}{d\tau}$, so one obtains

$$\dot{R}^2 = e^{-2}(1 - \frac{2GM}{R})^2(\frac{2GM}{R} - 1 + e^2),$$

with $e < 1$ for gravitationally bound matter. Plotting $\dot{R}^2$ vs R, there is a local zero at $R = 2GM$. On the other hand, using $\frac{d}{dt} = (dt/d\tau)^{-1}\frac{d}{d\tau}$,

$$\left(\frac{dR}{d\tau}\right)^2 = (1 - e^2)((R_{max}/R) - 1).$$

For a collapsing star, we only plot the diagram outside of the star. Looks like Minkowski space for early t, with a surface of the star plotted. Then get $r = 0$ singularity intersecting $\mathcal{I}^+$ inside of place where $r = 2GM$ goes from being a time-like surface to being a null surface. So $r = 0$ continues from being the vertical line of the Minkowski diagram into being the horizontal line at the top of the diagram, ending at the point (two-sphere) i_+ where it intersects $\mathcal{I}^+$. Note how for $r < 2GM$ the future horizon is the $r = 0$ singularity (ouch).

• Black holes vs Naked singularities. The $r = 0$ singularity in region II is behind the horizon. No signal from there can reach outside at future infinity, i.e. $\mathcal{I}^+$. On the other hand, the $r = 0$ singularity in region III can send signals to $\mathcal{I}^+$. That is called a naked singularity, not cloaked by a horizon.

The $M < 0$ Schwarzschild solution, which also solves Einstein's equations, also has a naked singularity at $r = 0$. It's Penrose diagram looks like Minkowski space-time, but with $r = 0$ an actual singularity. Light rays can get to $\mathcal{I}^+$ from $r = 0$, so it is a naked singularity.

Another example of a Penrose diagram with a naked singularity is a variant of that of a collapsing star, but where the $r = 0$ singularity at the top is drawn as a vertical extension of the $r = 0$ region of Minkowski for $t >$ some critical time, like patching together $M < 0$ Schwarzschild with Minkowski space.

Cosmic censorship conjecture "Naked singularities cannot form from gravitational collapse in an asymptotically flat space-time that is non-singular on some initial space like Cauchy hyper surface." Proving this conjecture is a major unsolved problem in GR. (In Carroll, includes the restriction that the dominant energy condition is satisfied i.e. the weak energy condition $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all time-like t^μ and the additional requirement that $T^{\mu\nu}t_\mu$ is non-space-like; for a perfect fluid, $\rho \geq |p|$ vs the WEC $\rho > 0$ and $\rho + p \geq 0$.)

• **Raychaudhuri's equation:** consider a bunch of nearby time-like geodesics, parameterized by proper time τ. Let $\xi^\mu(x)$ be unit tangent vectors to these geodesics, $\xi^\mu \xi_\mu = -1$. Then $B_{\mu\nu} = \nabla_\nu \xi_\mu$ satisfies $B_{\mu\nu}\xi^\mu = B_{\mu\nu}\xi^\nu = 0$, i.e. it is purely spatial. If η^μ is pictured as being orthogonal to the geodesic family, with $\mathcal{L}_\xi \eta^\mu = 0$, then it follows that $\xi^\mu \nabla_\mu \eta^\nu = \eta^\mu \nabla_\mu \xi^\nu = B^\mu{}_\nu \eta^\nu$. Let $h_{\mu\nu} \equiv g_{\mu\nu} + \xi_\mu \xi_\nu$, interpreted as the spatial metric. The expansion, shear, and twist of the congruence of geodesics are then

$$\theta \equiv B^{\mu\nu}h_{\mu\nu}, \quad \sigma_{\mu\nu} \equiv B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu}, \quad \omega_{\mu\nu} = B_{[\mu\nu]}.$$

Raychaudhuri's equation follows from taking $\xi^\lambda \nabla_\lambda B_{\mu\nu} = -B^\lambda{}_\nu B_{\mu\lambda} + R^\kappa_{\lambda\nu\mu}\xi^\lambda\xi_\kappa$ (which follows from the derivative of the geodesic equation, recalling geodesic deviation), and taking the trace:

$$\frac{d\theta}{d\tau} = \xi^\mu \nabla_\mu \theta = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}\xi^\mu\xi^\nu.$$

Using Einstein's equations the last term is $-8\pi G[T_{\mu\nu}\xi^\mu\xi^\nu + \frac{1}{2}T^\mu_\mu]$, and the strong energy condition is the (conjecture) that the term in square brackets is always non-negative. Then the last term in $d\theta/d\tau$ is negative (or zero), which can be interpreted as the attraction of gravity. If the geodesics are chosen such that $\omega_{\mu\nu} = 0$ then it follows that

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \rightarrow \frac{d}{d\tau}\theta^{-1} \geq \frac{1}{3}.$$

Then

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau.$$

If $\theta_0 < 0$, i.e. the congruence is initially converging, then θ^{-1} will go through zero in $\tau \leq 3/|\theta_0|$. This means there is a singularity in the congruence, like a caustic, but not necessarily a singularity in space-time. It can happen in Minkowski space. But it is a starting point for eventually proving singularity theorems.

• **Trapped surface:** a compact, two-dimensional, space like surface such that θ on both sets of geodesics (ingoing and outgoing) orthogonal to T are everywhere negative. The various singularity theorems connect trapped surfaces to singularities. E.g. in the Schwarzschild black hole in region II all surfaces are trapped.

• Black hole no hair (classically) theorem: stationary, asymptotically flat solutions of GR coupled to E&M that are nonsingular outside of the event horizon are fully specified by gauge charges (not global symmetry charges). The Schwarzschild black hole is specified by M, which is related to the energy. Other gauge charges are electric charge and angular momentum. The Reissner-Nordstrom (1918) black hole generalizes Schwarzschild to include electric charge. The Kerr black hole (1963) includes instead angular momentum. The Newmann et al solution (1965) includes both mass, electric charge, and angular momentum.