2/22/17 Lecture 12 outline

• Recall from last time: the Schwarzschild solution is the unique solution of Einstein's equations in vacuum with spherical symmetry:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2.$$

The object is a black hole of its radius has $R_{object} \leq R_S = 2GM/c^2$. For the sun, $GM_{sun}/c^2 = 1.48km$, so the sun would be a black hole if all its mass were compressed into a radius smaller than $\sim 3km$. A planet out at R_e could continue rotating around the black hole on the same orbit as if it were a normal star – things only get bizarre on distances $\sim R_s = 2GM/c^2$. As we said last time, Consider null, radial geodesics, see they have $dt/dr = \pm(1 - \frac{2GM}{r})$, so the slope of the light cones in the (r, t) plane close up at $r = R_S = 2GM$. A light ray just outside that radius seems to never get there, but this is an illusion of the coordinate system.

Let's sit well outside R_S and drop our friend C3P0 into the horizon, and he's going to send messages back to us. Give him $\ell = 0$, so $V_{eff}(r) = \frac{1}{2}\epsilon - \epsilon GM/r$ and $\epsilon = 1$. Take him initially at $r = r_0$ at $t = t_0$ Then we have $\frac{dt}{d\tau} = e(1 - 2GM/r)^{-1}$ and $\frac{dr}{d\tau} = -\sqrt{e^2 - 2V_{eff}(r)}$, where $e = p_\mu K^\mu$ is the conserved energy per unit mass associated with the time-like Killing vector. At $r = r_0$, $dt/d\tau = 1/\sqrt{1 - v_0^2} = 1$. So $e = 1 - 2GM/r_0$. The proper time to go from r_0 to r_1 is then

$$\Delta\tau_{C3P0} = \int_{r_0}^{r_1} \frac{dr}{\frac{dr}{d\tau}} = -\int_{r_0}^{r_1} \frac{dr}{\sqrt{e^2 - 1 + \frac{2GM}{r}}}.$$

The coordinate time is

$$\Delta t = \int_{r_0}^{r_1} d\tau \frac{dt}{d\tau} = -\int_{r_0}^{r_1} \frac{dr\ e}{(1 - \frac{2GM}{r})\sqrt{e^2 - 1 + \frac{2GM}{r}}}.$$

If we started him at $r \approx \infty$, then $e = 1$, and the proper time needed to go from r_1 to r_2 is

$$\Delta\tau_{C3PO} = -\int_{r_1}^{r_2} \sqrt{\frac{r}{2GM}} = \frac{2}{3\sqrt{2GM}} r^{3/2}|_{r_2}^{r_1}.$$

From any finite r_1, hit $r_2 = 2GM$ in a finite proper time, but coordinate time $\Delta t \to \infty$ for $r_2 \to R_S$. Also, the time that we see for the signals to get to us is $\Delta\tau_{US} \to \infty$. It seems to us that he never gets to the horizon.

• From the perspective of someone at $r = \infty$, the horizon is a special place. But from the perspective of someone falling in, nothing too special happens there. There are extreme tidal forces on length scales $\sim GM/c^2$, but if we imagine a supermassive black hole, this is a huge length scale and as long as the infalling observer is much smaller than that scale, they don't notice anything too special as they get too, and cross the horizon.

• Trade the original Schwarzschild time t for v, defined by $t = v - r - 2M \log |\frac{r}{2M} - 1|$, where we use $G = 1$ units. Then the metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the same Schwarzschild geometry, but in the Eddington-Finkelstein coordinates. The geometry and physics are unchanged, only the names of the coordinates have been altered to make the physics clearer.

Let's look at the radial light cones in this coordinate system: $-(1 - \frac{2M}{r})dv^2 + 2dvdr = 0$, so we can take v =constant, i.e. $\frac{dv}{dr} = 0$, which is an ingoing light ray since increasing t means decreasing r for v =constant. Another solution is $-(1 - \frac{2M}{r})dv + 2dr = 0$, so this null curve has $\frac{dv}{dr} = 2(1 - \frac{2GM}{r})^{-1}$, or integrating, $v - 2(r + 2GM \log |\frac{r}{2M} - 1|)$ =constant. This light ray is outgoing for $r > 2GM$, so it's the other side of the light cone. But for $r < 2GM$ it's also ingoing. Plot what's happening in the (r, v) plane. The entire light cone has been tilted, to point in toward the black hole. Uh-oh... no escape! And the horizon, $r = 2GM$, $v =$ constant, is actually a null surface. The fact that both light cones point to smaller r for $r < 2GM$ means that $r < 2GM$ surfaces are "trapped": since nothing can go faster than light, everything moves inward towards $r = 0$. We'll later mention singularity theorems, which connect trapped surfaces to singularities.

• Let $\widetilde{t} \equiv v - r$ and plot what happens for a collapsing star in the $(r, \widetilde{t})$ plane.

• Illustrate a fake, coordinate singularity: consider Minkowski space, restrict to $t > 0$. The space $t < 0$ is still there, but we'll pretend that the coordinate t doesn't cover it anymore, it just ends. But the space is fine, we just need to continue past where our bad coordinate ends. Can make it more obscure, but still the same story, by taking $t \to 1/t$ now, get $ds^2 = -dt^2/t^4 + dx^2$.

Another example that you've seen in a HW is Rindler space, $ds^2 = -x^2dt^2 + dx^2$. Geodesics end in a finite proper time at $x = 0$. Is that a bad point? No: by a coordinate transformation $ds^2 = -dT^2 + dX^2$, it's just Minkowski space, and the original spacetime is in the wedge $|X| > T$. Just need to continue to all X and T, and no problem.

• Schwarzschild black hole in Eddington Finkelstein coordinates, $t = v - r - 2GM \log|\frac{r}{2GM} - 1|$. Then the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Schwarzschild time t ends at the horizon, but the coordinate v keeps on going, just like the proper time of an infalling observer. (Proper time is coordinate independent.)

The horizon, $r = 2GM$, $v =$ constant, is a null surface.

• Let $\widetilde{t} \equiv v - r$ and plot what happens for a collapsing star in the $(r, \widetilde{t})$ plane.

An outside observer sees the collapse only asymptotically, and the escaping light becomes increasingly redshifted. Consider light emitted at radius r_E, and received at a distant radius r_R and at time t_R, get $-4M\log(\frac{r_E}{2M} - 1) \approx t_R - r_R$. Using $\omega_R = -(u_{obs})_\mu k^\mu_\gamma$ and $\omega_E = -(u_{star})_\mu k^\mu_\gamma$, get $\omega_R \sim \omega_E e^{-t_R/4M}$.

• Matter inside $r = 2GM$ necessarily hits $r = 0$, and it happens in finite proper time,

$$\Delta\tau = \frac{1}{\sqrt{2GM}} \frac{2}{3} r^{3/2}|_0^{2GM} = \frac{2}{3}(2GM).$$

• Kruskal coordinates: $X^2 - T^2 > -1$ and

$$(-1 + r/2GM)e^{r/2GM} = X^2 - T^2, \qquad t/2GM = \ln(\frac{T+X}{X-T}).$$

$$ds^2 = 32\frac{M^3 e^{r/2GM}}{r}(-dT^2 + dX^2) + r^2 d^2\Omega_2.$$

Regions I,II, III, IV in (T, X) plane.

• The corresponding Penrose diagram can be found on the cover of Townsend's notes. Again regions I, II, III, IV. In region 1, $r > 2GM$, the region outside of some $r > r_0 > 2GM$ looks similar to Minkowski space, with $r = r_0$ intersecting i_- and i^+ in the infinite past and future, null boundaries $\mathcal{I}^-$ and $\mathcal{I}^+$, which meet at i_0. Note that observers at e.g. $\mathcal{I}^+$ don't see any difference from Minkowski space, and can't see the (null) horizon. The difference from Minkowski space is that there is the additional region II.

For $0 < r < 2GM$, in region II, it looks a bit like the Penrose diagram of Minkowski space, but turned on its side. There is a space like singularity at $r = 0$ that meets i_+ at the end of the null horizon at $r = 2GM$. Regions III and IV are white-hole copies of regions II and I. All $r =$ constant hyper surfaces end at i_+.

Pick a hypersurface of constant time, e.g. $T = 0$, and draw an embedding diagram of the curvature. Get a wormhole tube connecting two exterior regions. This is called the

Einstein Rosen (1935) bridge, discovered by Ludwig Flamm in 1916. In 1962 Wheeler and Robert Fuller showed it is unstable and non-traversable. Using exotic hypothetical matter (e.g. violating various of the positive energy conditions) one can (possibly) find traversable wormholes.

• Now discuss non-eternal black holes. Then regions III and IV are not present.

Brief detour on solving Einstein's equations in the star: take $T_{\mu\nu}$ to be the perfect fluid form, with pressure $p = p(r)$ and energy density $\rho = \rho(r)$. The metric form is spherically symmetric and feeds back into the r dependence of the pressure and density via the stress-tensor conservation equation. This leads to the Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium (using $G = 1$ units)

$$\frac{dp}{dr} = -(p+\rho)\frac{m(r) + 4\pi r^2 p}{r(r-2m(r))}.$$

For $p \ll \rho$, $m(r) \ll r$ it reduces to the Newtonian equilibrium equation. Qualitatively, find that for fixed ρ the pressure p is greater in GR than in Newtonian theory. E.g. for $\rho = \rho_0$ a constant get $p^{Newtonian}(r) = \frac{2}{3}\pi\rho_0^2(R^2 - r^2)$ and (Schwarzschild 1916)

$$p^{GR}(r) = \rho_0 \left[\frac{\sqrt{1-2GM/R} - \sqrt{1-2GMr^2/R^3}}{\sqrt{1-2GMr^2/R^3} - 3\sqrt{1-2GM/R}}\right].$$

Note the pressure at $r = 0$ becomes infinite if $M \geq 4R/9G$: such stars cannot exist in GR. Instead, they collapse to a black hole. Can plot M vs R and see that, as R decreases, the central ρ_c goes up to a maximum (white dwarfs, Chandrsekhar maximum, 1939), then it becomes unstable and M decreases with decreasing R until the local minimum where neutron stars form. Then the density and M goes back to increasing with decreasing R until the point where black holes form.

• Consider dust (pressureless matter) falling in Schwarzschild geometry: the geodesic equation gives $r(\tau) = (3/2)^{2/3}(2GM)^{1/3}(\tau_* - \tau)^{2/3}$. Continue via $v(r)$ to $r < 2GM$ and find, once horizon is passed, matter hits $r = 0$ in proper time $\Delta\tau = 4GM/3$.