2/15/17 Lecture 11 outline

• Recall from last time: the Schwarzschild solution is the unique solution of Einstein's equations in vacuum with spherical symmetry:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

The Ricci tensor vanishes for Schwarzschild, but the Riemann tensor does not, e.g.

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}.$$

We see that $r = 0$ is really a singularity whereas $r = R_s$ is not a real singularity.

The fact that the metric is static and spherically symmetric implies corresponding Killing vectors. For objects on geodesics, the 4-momentum contracted with the Killing vectors leads to the conserved energy and angular momentum.

$K^\mu = \delta^\mu_t = (1, 0, 0, 0)$. Get

$$H \to \partial_t \to K_\mu = (-(1 - \frac{2GM}{r}), 0, 0, 0).$$

$$L_z \to \partial_\phi \to L_\mu = (0, 0, 0, r^2 \sin^2\theta).$$

Taking $\theta = \pi/2$, the conserved quantities $p_\mu V^\mu$ are

$$e = (1 - \frac{2GM}{r})\frac{dt}{d\lambda}, \qquad \ell = r^2\frac{d\phi}{d\lambda}.$$

For massive particles, these are actually E/m and L_z/m.

• If the observer is stationary only $u^0_{obs} \neq 0$, and is given by $-g_{00}(u^0_{obs})^2 = -1$, so $u^\mu_{obs} = (-g_{00})^{-1/2}K^\mu$, and thus $\widehat{E}_{obj;obs} = e_{obj}(-g_{00})^{-1/2}$. Now e_{obj} is a constant, but g_{00} depends on r, so a stationary observer at fixed r measures $E_{obj,obs}$ depending on r. At $r \to \infty$, $g_{00} \to -1$, so $e_{obj} = E_{obj,\infty}$, the energy observed by an observer at infinity. To summarize,

$$E_{obj,obs}(r) = E_{obj,\infty}(1 - \frac{2GM}{r})^{-1/2}, \qquad \widehat{E}_{obj,\infty} \equiv e$$

This applies whether the observer is measuring the energy of an apple or a photon. For photons, it implies the gravitational redshift

$$\omega_\gamma(r) = \omega_\gamma(\infty)(1 - \frac{2GM}{r})^{-1/2}.$$

Therefore, a stationary observer at radius r_1 will measure the photon as having frequency $\omega_1 = \omega(r_1)$ and another at radius r_2 will measure it as having frequency $\omega_2 = \omega(r_2)$, with

$$\frac{\omega_2}{\omega_1} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2}\right)^{1/2}.$$

E.g. for $r \gg 2GM$ this gives $\omega_2/\omega_1 \approx 1 + \Phi_1 - \Phi_2$, which is what we saw before from the rocket picture. This formula makes sense for $r > 2GM$. A photon starting at $r = 2GM$ would be redshifted to zero frequency by the time it gets to infinity – in other words, it can't make it out.

We can also use this to determine the escape velocity needed for a massive object, starting at fixed r, to make it to infinity. (For massless objects, there's no notion of escape velocity – it can always make it to infinity from any $r > 2GM$. For $r \leq 2GM$, the light doesn't escape, as we saw from the redshift formula. To make it to infinity, need $e = 1$, so the observer at fixed r needs to see the object as having

$$\widehat{E}_{obj,obs,esc} = (1 - \frac{2GM}{r})^{-1/2} \equiv (1 - V_{esc}^2/c^2)^{-1/2},$$

so $V_{esc} = \sqrt{2M/R}$, coincidentally the same as in Newtonian mechanics. For $r \to 2GM$, get $V_{esc} \to c$.

• Back to the geodesic equations. Use constants e, ℓ, and also $u \cdot u = -\epsilon$, where $\epsilon \equiv 1$ for massive objects and $\epsilon \equiv 0$ for massless ones. Conservation of angular momentum implies that orbits lie in a plane. E.g. if the particle is moving with $d\phi/d\tau$ at an instant, then $\ell = 0$ for all time. Instead take $\theta = \pi/2$ and $u^\theta = 0$, and it remains so for all time.

The radial geodesic equation leads to

$$\tfrac{1}{2}(\frac{dr}{d\lambda})^2 + V_{eff}(r) = \tfrac{1}{2}E^2,$$

$$V_{eff}(r) = \tfrac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3},$$

with $\epsilon \equiv -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ so $\epsilon = 1$ for a massive particle with $\lambda = \tau$ and $\epsilon = 0$ for a massless particle. For a massive object we can multiply the above by m and use $L = \ell m$ to make the first two terms look familiar. The first term is the Newtonian potential, there only for massive objects. The second term is the angular momentum barrier, there for both massive and massless objects. The third term has $\gamma_{GR} = 1$, and $\gamma_{Newtonian} = 0$;since its $\sim 1/r^3$ its negligible away from the origin but it dominates for sufficiently small r. It replaces the infinite centrifugal barrier of Newtonian mechanics with a barrier of finite height.

• Draw pictures for timelike (massive) and null (massless) cases, compare / contrast with Newtonian case. For a massive object, the shape of $V_{eff}(r)$ depends on the size of ℓ. For a massless object, ℓ affects only the overall scale size of $V_{eff}(r)$, not its shape.

• Look for circular orbits, $dV/dr = 0$: $\epsilon MGr_c^2 - \ell^2 r_c + 3GML^2\gamma = 0$. For massless case, $\epsilon = 0$, no solution for $\gamma = 0$, but for $\gamma = 1$ get $r_c = 3GM$. This is a local maximum, unstable to perturbations. For the massive case, $\epsilon = 1$, get

$$r_c = \frac{\ell^2 \pm \sqrt{\ell^4 - 12GM^2\ell^2}}{2GM}.$$

For $\ell^2 > 12GM^2$, the inner one is unstable and the outer one is stable. For $\ell \gg 1$ get $r_c \approx \ell^2/GM$, which is the stable Newtonian result, and $r_c = 3GM$, which is unstable.

For $\ell^2 = 12GM^2$, there is only 1 orbit, at $r_c = 6GM$. This is the smallest possible stable orbit. For $\ell^2 < 12GM^2$, there are no extrema of V_{eff}, the potential just slides down, down, down to the singularity at $r = 0$, goodbye.

• Consider the null case. The minimum e needed to climb the barrier is given by $\frac{1}{2}e^2 = V_{eff}(r = 3GM) = \ell^2/2(27)(GM)^2$, or $\ell^2/e^2 = 27(GM)^2$. At infinity, we have $\ell = be$, where b is the impact parameter. To see that note that, at infinity, $\ell/e = r^2\frac{d\phi}{d\lambda}/(1 - \frac{2GM}{r})\frac{dt}{d\lambda} \to r^2 d\phi/dt$ and so $\phi \approx b/r$, with $dr/dt \approx -1$. So we see that light with impact parameter less than $b_c = 3^{3/2}GM$ is captured. The capture cross section is $\sigma_c = 27\pi(GM)^2$.

• Study precession of the perihelion + deflection of light, multiplying the radial equation by $(d\lambda/d\phi)^2 = r^4/\ell^2$ to convert $(dr/d\lambda)^2$ there into $(dr/d\phi)^2$:

$$(\frac{dr}{d\phi})^2 + 2\frac{r^4}{\ell^2}V_{eff}(r) = \frac{r^4}{\ell^2}e^2.$$

So

$$\Delta\phi = \int dr \frac{d\phi}{dr} = \int dr \frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V_{eff}(r)}}.$$

$$V_{eff}(r) = \tfrac{1}{2}\epsilon - \frac{\epsilon GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\gamma\ell^2}{r^3}.$$

For an orbit, between the inner and outer turning points (the zeros of $e^2 = 2V(r)$), get

$$\Delta\phi = 2\int_{r_1}^{r_2} dr \frac{\ell}{r^2}\frac{1}{\sqrt{e^2 - 2V(r)}}.$$

In the Newtonian case, $\gamma = 0$, can do the integral, get $\Delta\phi = 2\pi$, so the orbits come back to themselves. For $\gamma = 1$, get $\Delta\phi > 2\pi$, so they overclose, and the perihelion precesses.

Let $x = \ell^2/GMr$, in the equation we're integrating above, and take $\frac{d}{d\varphi}$ of that equation to obtain

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2\gamma x^2}{\ell^2}.$$

If $GM/\ell \ll 1$, we can treat the last term as a perturbation, $x = x_0 + x_1$, with $x_0 = 1 + \xi \cos\phi$, where $\xi^2 = 1 - b^2/a^2$ is the eccentricity of the ellipse, usually called e. Then find $x \approx 1 + \xi\cos(1-\alpha)\phi$, and $\Delta\phi \approx 2\pi(1+\alpha)$, where $2\pi\alpha \approx 6\pi G^2M^2/\ell^2 \approx 6\pi GM/c^2a(1-\xi^2)$. For Mercury, use $GM_{sun}/c^2 = 1/48 \times 10^5 cm$, $a = 5.79 \times 10^{12} cm$, $\xi = 0.2056$, to get $2\pi\alpha \approx 5.01 \times 10^{-7}$ radians per orbit, which had been observed before (!) the GR calculation.

Deflection of light (radially, comes in and bounces off the $V_{eff}(r)$ barrier):

$$\Delta\phi = 2\ell \int_{r_1}^{\infty} \frac{dr}{r^2} \frac{1}{\sqrt{e^2 - 2V(r)}}, \qquad V(r) = \frac{\ell^2}{2r^2} - \frac{\gamma GM\ell^2}{r^3}$$

Using $\ell/e = b$ and defining $w = b/r$ get

$$\Delta\phi = 2\int_0^{w_1} dw (1 - w^2(1 - \frac{2GMw\gamma}{b}))^{-1/2}.$$

For $\gamma = 0$, get $\Delta\phi = \pi$, no deflection in Newtonian case. For $\gamma = 1$, get $\Delta\phi > \pi$, corresponding to focusing. Approximating $GM/b \ll 1$, get $\Delta\phi \approx \pi + \frac{4GM}{b}$. Deflection becomes infinite for $GM/b \to 1/\sqrt{27}$, which is where we already saw last time is the b_{crit} needed to overcome the barrier in $V_{eff}(r)$ (at $r = 3GM$). Also time delay of light in GR.