2/13/17 Lecture 10 outline

- Recall from last time: Friedmann Robertson Walker space times.

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2,$$

where the 3d space $d\Sigma^2$ is maximally symmetric. Again, three possibilities: the 3d space can have $k = R_{3d}/6$ negative (open), positive (flat), or positive (closed). By a choice of coordinates,

$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

with $k = 0, 1, -1$. Or

$$d\Sigma^2 = d\chi^2 + f(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with $f(\chi) = \sin\chi, \chi, \sinh\chi$, respectively, for $k = 1, 0, -1$. The χ ranges are $\chi_{k=0,-1} \in [0, \infty]$ and $\chi_{k=1} \in [0, \pi]$. The $k = 0, -1$ cases are infinite, topologically R^3, while the $k = 1$ case is closed, topologically S^3.

The symmetry of the RW space times require that the energy-momentum tensor be that of a perfect fluid: $T_{\mu\nu} = (p+\rho)U_\mu U_\nu + pg_{\mu\nu}$. Conservation of energy requires $\dot\rho/\rho = -3(1+w)\dot a/a$, where $w \equiv p/\rho$. For constant w this gives $\rho \sim a^{-3(1+w)}$.

Recall e.g. that the null dominant energy condition conjecture is $|w| \leq 1$. Einstein's equations ($G_{\mu\nu}u^\mu u^\nu = 8\pi G\rho$ and $G_{\mu\nu}s^\mu s^\nu = 8\pi Gp$) lead to the Friedmann equations:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

For $a \neq 0$, the first equation can be obtained as the integral of the second one. It gives the constant of integration as being equal to the same constant k.

As found by Einstein, if ρ and p are non-negative, it is impossible to have a constant. He introduced a cosmological constant component, which has $p_\Lambda = -\rho_\Lambda$, to get constant a, which he later referred to as his greatest blunder (though he turned out to be right about $\Lambda \neq 0$).

Three simple cases (universe is an admixture of these and possibly others): $\rho \sim a^{-n}$ with equation of state $w = \frac{1}{3}n - 1$. Matter has $n = 3$ (so $w = 0$), radiation has $n = 4$ (so $w = 1/3$), curvature has $n = 2$ (so $w = -1/3$), and vacuum has $n = 0$, so $w = -1$.

For example, the Einstein static universe is a solution with $\rho_\Lambda = \frac{1}{2}\rho_M$; it is topologically $R \times S^3$.

For $k = 0$: in matter-dominated era, $a(t) = (t/t_0)^{2/3}$. For a radiation dominated era, $a(t) = (t/t_0)^{1/2}$. For a vacuum dominated era, $a(t) = e^{H(t-t_0)}$, with $H^2 \equiv 8\pi G\rho_v/3 \equiv \Lambda/3$.

It follows from the energy conservation equation that ρ decreases as the universe expands, and was higher in the past. As $a \to 0$, $\rho \to \infty$, so $a \to 0$ is a physical singularity, not just a harmless coordinate singularity. As $a \to 0$, space-time is singular, and Einstein's equations must break down before then, e.g. quantum effects must kick in. The past singularity "big bang" is there for all cases with $\rho + 3p > 0$. It can be evaded by e..g a positive Λ. For ρ and p non-negative, one can ask if the singularity can be evaded by a non-spherically symmetric configuration. Hawking proved in his PhD thesis that the singularity is still there, with fewer and fewer assumptions: singularity theorems. Will touch on them more later.

The Hubble parameter $H \equiv \dot{a}(t)$ is currently the Hubble constant $H_0 = H(t_0)$. $H(t_0)^{-1} \approx 9.78^{-1}h^{-1} \times 10^9$ years, with $h \approx .72$. Let $\rho_{crit} \equiv 3H_0^2/8\pi G \equiv 1.99 \times 10^{-29}h^2 g/cm^3$. Define $\Omega_{m,r,v} \equiv \rho_{m,r,v}/\rho_{crit}$. Matter has $p_m \approx 0$, radiation (blackbody spectrum) has $p_r = \rho_r/3$, and vacuum CC has $p_v = -\rho_v$. If $\Omega = \Omega_v + \Omega_r + \Omega_v = 1$, then $k = 0$ and the universe is flat. This is what observation suggests to be the case in our universe: $\Omega_m \approx 4.6\%$, $\Omega_{d.m.} \approx 24\%$, $\Omega_v \approx 71.4\%$. The scaling of $\rho(t)$ is such that radiation dominated for $t \to 0$, then matter, and finally vacuum.

For $k = 0$ and $k = -1$, and $\rho > 0$, note that $\dot{a} > 0$ so the universe will expand forever. For any matter with $p > 0$, ρ must decrease as a increases at least as rapidly as a^{-3}, so $\rho a^2 \to 0$ as $a \to \infty$. For $k = 0$ the expansion velocity $\dot{a} \to 0$ as $\tau \to \infty$, and for $k = -1$, $\dot{a} \to 1$. For $k = 1$, the universe cannot expand forever: eventually RHS wants to become negative, but the LHS is positive, so $a \leq a_{crit}$ and this happens for finite t. There is a bounce, where $a \to a_{crit}$ and then the universe re-contracts. A finite t after the big bang, $a \to 0$ again, in a big crunch. The spatially closed 3-sphere universe will only exist a finite span of time.

• Cosmological redshift. At event P_1, at time t_1, a photon is emitted with frequency ω_1. It is then observed at event P_2 at time t_2. Let's find ω_2. Recall that the frequency of light measured by an observer with 4-velocity u^μ is $\omega = -k_\mu u^\mu$. In flat spacetime, a stationary observer has $u^\mu = \delta^\mu_t$, with $u_\mu u^\mu = -1$. In a static spacetime, a stationary observer has $u^\mu = K^\mu/\sqrt{-K_\mu K^\mu}$, where K^μ is the time-like Killing vector. Recall that, for geodesics, $p_\mu K^\mu$ is a constant of the motion, where p_μ is an object's 4-momentum and K^μ

is any Killing vector. So the light has $k_\mu K^\mu$ as a constant of the motion, as it moves along a null geodesic in the spacetime. So $\omega_2/\omega_1 = \sqrt{(-K\cdot K)_2/(-K\cdot K)_1} = a(\tau_2)/a(\tau_1)$. The wavelength expands with the universe, which makes sense. So $z \equiv (\lambda_2 - \lambda_1)/\lambda_1 = \frac{a(\tau_2)}{a(\tau_1)} - 1$.

• Penrose diagrams for FRW solutions. Define τ by $\tau = \int dt/a(t)$. Then

$$ds^2 = a^2(\tau)(-d\tau^2 + d\chi^2 + f(\chi)^2 d\Omega_2).$$

Depending on $k = 0, \pm 1$, this conformally maps FRW to Minkowski, de Sitter, or anti-de Sitter. However, we have to account for the fact that $t \geq 0$, with $a \to 0$ at $t = 0$. Is there a past horizon for event P? The Penrose diagram has τ instead of t, so the issue is whether, as $t \to 0$, is it the case that $\tau \to \tau_0$ finite, or $\tau \to -\infty$. Observer there can receive a signal from all other observers iff $\tau = \int dt/a(t)$ diverges as $a \to 0$. Suppose that $a(t) \propto t^q$: the integral diverges for $t \to 0$ if $q \geq 0$, and it converges for $0 < q < 1$.

Consider e.g. for $k = 0$, so conformally related to flat Minkowski space-time, and recall that $q_{matter} = 2/3$ and $q_{radiation} = 1/2$, so in the $0 < q < 1$ range where the integral converges: $\tau \to \tau_c$ finite as $t \to 0$. So there is a space like singularity $\mathcal{I}^-$ at $t = 0$, where $a \to 0$, which has an point i^0 (really an S^2) corresponding to spatial infinity. Looks like Minkowski space diagram, cut in half and keeping only the upper half. A past horizon. The term involving k is negligible for $a \to 0$, so they will have similar past horizons. For e.g. $k = 1$ with mostly dust, the particle horizon ceases to exist at as $a \to a_c$. A light ray emitted at the big bang would travel halfway around the S^3 by the moment of maximal expansion. For $k = 1$, the above is already the Einstein static universe and, in addition to the past horizon there is also the future horizon at $t = \pi/2$. For $p = \Lambda = 0$, $\tau \in (0, \pi)$, so it is the same as de Sitter. For $p > 0$, instead $\tau \in (0, \tau_+)$ for some $\tau_+ < \pi$. For $k = -1$, there is a past space like $\mathcal{I}^-$ at $t' = 0$.

• Finding symmetric solutions of Einstein's equations, continued. Assume stationary and spherically symmetric. So assume Killing vectors for time translations, $K^\mu = \delta^\mu_t$, and rotations L^μ_θ and L^μ_ϕ. Recall that Killing vectors satisfy $\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\nu = 0$. Up to coordinate transformations, the general metric with these symmetries are of the form

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2, \qquad (1)$$

E.g. $e^{2\gamma(r)}r^2 d\Omega^2 \to r^2 d\Omega^2$ via $r \to e^\gamma r$. Consider solving Einstein's equations for $T^{\mu\nu} = 0$, e.g. in the region outside of a star. The unique solution is the Schwarzschild solution. Birkoff's theorem shows that, for $T^{\mu\nu} = 0$, if we assume spherical symmetry but do not assume static, the unique solution of Einstein's equation turns out to also be static: static comes for free, if it's vacuum and spherically symmetric.

Compute $R_{\mu\nu}$ and see that it vanishes only if $\alpha = -\beta$ and $\partial_r(re^{2\alpha}) = 1$, which gives the Schwarzschild solution, $e^{2\alpha} = 1 - R_s/r$. Recall that we know from the Newtonian limit that $h_{00} = 2\Phi$, so $R_s = 2GM$:

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

The Ricci tensor vanishes for Schwarzschild, but the Riemann tensor does not. Write out some example components, e.g. $R^r_{\phi r\phi} = re^{-2\beta}\sin^2\theta\partial_r\beta$, $R^t_{\theta t\theta} = -GM/r$, etc. The non-zero Riemann tensor will give e.g. the correct focusing of nearby geodesics,

$$\frac{D^2}{d\lambda^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\delta x^\sigma.$$

Also,

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}.$$

We see that $r = 0$ is really a singularity whereas $r = R_s$ is not a real singularity.