1/9/17 Lecture 1 outline

• Introduction and lightning review. Start by comparing and contrasting gravity vs the other known fundamental forces (electromagnetism, strong, weak).

0. Trivial (but annoying): standard convention for the metric signature. I like mostly minus (so $p^2 = m^2$) whereas it is standard (and useful) in GR to use mostly plus.

1. At experimentally accessible short scales, gravity is by far the weakest force. Compare gravity to E&M at e.g. atomic scales for a Hydrogen atom: $\sim Gm_1m_2/e^2 \sim m_1m_2M_{pl}^{-2}/\alpha \sim 10^{-42}$ where $\alpha \approx 1/137$ is the fine structure constant and $M_{pl} \sim 10^{19}GeV \sim 10^{-8}kg$ is the Planck mass. But on larger scales it is the most important force. Because other forces have $+$ and $-$ charges, leading to neutral objects and then remaining forces are merely weaker higher multipole moments. But gravity affects all masses positively, so no gravity neutrals, and the effects compound at larger and larger scales. Even light is bent by gravity so, since nothing can travel faster than light, gravity affects the casual structure of space-time.

2. All forces are based on local symmetry gauge invariances. For the non-gravity forces these are $SU(3)_C \times SU(2)_W \times U(1)_Y$ and at low-energy this becomes $U(1)_{EM}$. The symmetry acts as $\psi_q \to e^{iqf(x)/\hbar c}\psi_q$, where ψ_q is a quantum-mechanical wavefunction, or quantum field theory field, of charge q. The QED Lagrangian density, for example, is
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\text{matter}}(\psi, D_\mu^{(q)}\psi) + \mathcal{L}_{\text{other}} \supset -A_\mu J^\mu$$
where $D_\mu^{(q)} = \partial_\mu + iqA_\mu$ is the covariant derivative.

The $\mathcal{L}_{\text{other}}$ term is separated out to make a point: there are generally also terms with $F_{\mu\nu}$ appearing in other ways. For example, quantum effects can induce terms $\sim F^4$ and there can also be interactions $\mathcal{O}_{\mu\nu}F^{\mu\nu}$ with $\mathcal{O}_{\mu\nu}$ involving the matter fields, e.g. the magnetic moment interactions that are induced by quantum loops. (Their analog in GR, to be mentioned soon, could be present and lead to small deviations from the strongest form of the equivalence principle.)

Under the gauge transformation (setting $\hbar = c = 1$ for convenience here)
$$\psi_q \to e^{iqf(x)}\psi_q(x), \qquad A_\mu \to A_\mu + \partial_\mu f(x), \qquad \text{so} \qquad D_\mu^{(q)}\psi_q(x) \to e^{iqf(x)}D_\mu^{(q)}\psi_q(x).$$

The gauge field A_μ and covariant derivative are needed so that we can form derivatives that transform nicely. Gauge symmetry requires that $\mathcal{L}$ is invariant under these transformations. Gauge transformations are not observable – the whole point is that physics does not care about such transformations. Hence they are not really symmetries in the global (as in spatially independent) symmetry sense: instead they have to do with eliminating unphysical degrees of freedom associated with a useful redundancy. Note that gauge invariance requires current conservation, $\partial_\mu J^\mu = 0$, since varying $\mathcal{L}$ by a total derivative does nothing.

These theories have a global symmetry invariance under the Poincare group of translations, rotations, and boosts, e.g. $x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$. Correspondingly, there is a conserved energy-momentum tensor $T^{\mu\nu}$, with $\partial_\mu T^{\mu\nu} = 0$, that is constructed from a^μ shift invariance via Noether's procedure.

GR, on the other hand, is based on promoting Poincare symmetry to a gauge symmetry under general coordinate transformations $x^\mu \to x^{\mu'}(x)$. Scalars, vectors, tensors, transform nicely e.g. :

$$V^{\mu'}{}_{\nu'} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\mu'}} V^\mu{}_\nu, \qquad \nabla_{\lambda'} V^{\mu'}_{\nu'} = \frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\mu'}} \nabla_\lambda V^\mu{}_\nu.$$

The covariant derivatives require a connection, in order to yield nice tensors after taking the derivative, e.g.

$$\nabla_\rho V_\mu = \partial_\rho V_\mu - \Gamma^\lambda_{\rho\mu} V_\lambda, \qquad \nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\sigma} V^\sigma.$$

The opposite signs ensure that e.g. $\nabla_\rho(V_\mu V^\mu) = \partial_\rho(V_\mu V^\mu)$.

Local theories with this symmetry have the space-time metric $g_{\mu\nu}(x)$, which transforms as a tensor, such that $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is invariant. The metric $g_{\mu\nu}$ is the analog of A_μ in electromagnetism – it is the basic, dynamical field variable, though not directly observable according to gauge symmetry, and it is determined as a solution of a differential equation by the equations of motion.

The condition that $g_{\mu\nu} v^\mu w^\nu$ is unchanged under parallel transport requires $\nabla_\lambda g_{\mu\nu} = 0$. This is (torsion free) coordinate basis, and in this basis the connection can be written in terms of the metric

$$\Gamma^\mu_{\rho\sigma} = \tfrac{1}{2} g^{\mu\lambda}(\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma}).$$

The usual equations of flat space must be replaced with covariant versions, e.g. charge conservation becomes

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\sigma} J^\sigma = 0.$$

Conservation of energy and momentum becomes

$$\nabla_\mu T^{\mu\sigma} = \partial_\mu T^{\mu\sigma} + \Gamma^\mu_{\mu\lambda} T^{\lambda\sigma} + \Gamma^\sigma_{\mu\lambda} T^{\mu\lambda} = 0.$$

The extra terms have a nice interpretation in terms of Stokes', Gauss theorem. Letting $|g| \equiv= -\det_{\mu\nu}(g_{\mu\nu})$, the scalar integration measure is $\sqrt{|g|}d^4x$, where the $\sqrt{|g|}$ just cancels the Jacobian determinant. So $Q_{encl} = \int \sqrt{|g|}d^3x J^0$ is the scalar, conserved charge. Since $\Gamma^\mu_{\mu\sigma} = \frac{1}{\sqrt{|g|}}\partial_\sigma \sqrt{|g|}$, $\nabla_\mu J^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}J^\mu)$.

3. Quantum differences: The non-gravity forces are communicated by spin 1 messengers: the photon, and its analogs for the weak and strong forces (W-bosons, Z-bosons, and gluons). The graviton is a ripple of $g_{\mu\nu}$ and hence has spin 2. Moreover, in $\hbar = c = 1$, the other forces have a strength that is classically dimensionless, e.g. $\alpha_{EM} \approx 1/137$ whereas gravity has $G_N \sim 1/M_{pl}^2$. So quantum effects in perturbation theory would go like powers of $G_N E^2$, and are irrelevant (in the QFT technical sense, and in everyday use sense) for $E \ll M_{pl}^2$, as is the case at the LHC etc. On the other hand, these effects become large for $E \sim M_{pl}$, so the theory must be UV completed to something else at or before the Planck scale. String theory is the most popular candidate.

4. GR and the equivalence principle says that gravity can be thought of as not a force but as, instead, free-fall in the curved space-time metric. A free-falling particle has world line $x^\mu(\lambda)$ that satisfies

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0 \rightarrow \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.$$

Recall that $\frac{D}{d\lambda} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma$. The geodesic equation follows from extremizing the action:

$$I \sim \int (-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda})^{1/2} d\lambda \qquad \text{or} \qquad I \sim \int g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda$$

under $x^\mu \to x^\mu + \delta x^\mu$ (so also $g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}\delta x^\sigma)$).

Other forces contribute to an acceleration term on the RHS of the geodesic equation, e.g. E and M for a charged q particle of mass m lead to

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = \frac{q}{m}F^\mu{}_\nu \frac{dx^\nu}{d\tau}.$$

(Aside: in Kaluza-Klein theory, there is a 5th dimension, a circle, and this can be understood as coming from GR in 5d, with q related to the momentum around S^1, and the above term on the RHS can be moved to the LHS and reinterpreted in terms of $D^2x^\mu/d\tau^2 = 0$.

5. Maxwell's equations follow from varying

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\rm matter}(\psi, D_\mu^{(q)}\psi) + \mathcal{L}_{\rm other} \supset -A_\mu J^\mu$$

with respect to δA_μ. Likewise, Einstein's equations follow from varying the Einstein-Hilbert action with respect to $\delta g_{\mu\nu}$:

$$S = \int d^dx\sqrt{|g|}\left[\frac{1}{16\pi G}R + \mathcal{L}_{\rm matter}(\eta \to g, \partial_\mu \to \nabla_\mu) + \mathcal{L}_{\rm other}\right].$$

Here $\mathcal{L}_{\rm other}$ can be other terms involving curvature and the other fields, e.g. maybe a term $\sim R^4$ and / or a term $\sim RF_{\mu\nu}F^{\mu\nu}$. Einstein's GR and the strongest form of the equivalence principle are the *conjecture* that such terms are absent. This conjecture has not yet been tested to sufficient accuracy to determine if it is correct or false – it would not ruin the rest of the theory structure if $\mathcal{L}_{\rm other} \neq 0$, and indeed such terms can generally be induced by quantum loops, and they are present in string theory and M-theory etc. For the moment, we often assume $\mathcal{L}_{\rm other} \neq 0$. Dimensional analysis: $\mathcal{L} = d$, $[g] = 0$, $[R] = 2$, thus $[G] = 2 - d$, i.e. $G \sim 1/M_{pl}^{d-2}$. Higher order R terms might be expected to depend on the dimensionless quantity $R/M_{pl}^2 \sim R\ell_{pl}^2$, which is negligible unless the curvature radius is on order of $\ell_{pl} \sim 10^{-33}cm$ (!).

The $g_{\mu\nu}$ Euler Lagrange equations then give

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad T_{\mu\nu} = -2\frac{1}{\sqrt{|g|}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}.$$

The last expression for $T_{\mu\nu}$ is equivalent to that found via Noether's procedure. The $\mathcal{L}_{\rm other}$ possible terms will lead to variants of (deviations from) Einstein's equations.