Physics 225b, Homework 4, Due Wednesday Feb 15.

Exercises all based on those in Hartle, chapter 18.

1. Verify that the Friedmann equation is solved, in the case of a matter dominated, closed ($k = 1$) FRW model with $\Omega = \Omega_m > 1$, $\Omega_r = \Omega_v = 0$, by

$$a(\eta) = \frac{\Omega}{2H_0(\Omega - 1)^{3/2}}(1 - \cos\eta), \quad t(\eta) = \frac{\Omega}{2H_0(\Omega - 1)^{3/2}}(\eta - \sin\eta).$$

   Note that a expands from a big bang singularity at $a = 0$, corresponding to $\eta = 0$, to a maximal volume at $\eta = \pi$, and then re-collapses to a big crunch singularity at $\eta = 2\pi$. Note that their total duration is thus $(\pi\Omega/2H_0)(\Omega - 1)^{-3/2}$.

2. (a) Show that if the parameter η of the previous question is used as a time coordinate then

$$ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)).$$

   (b) Draw a η, χ spacetime diagram indicating the big bang, big crunch, and the past light cone of a comoving observer at the moment of maximum expansion.

   (c) Is there time before the big crunch for an observer to receive information from all parts of this spatially finite universe, or are there parts of it that he or she or it are doomed to never see?

   (d) Could an observer traverse the entire circumference of the universe in the time between the big bang and big crunch?

3. Show that for any FRW theory with matter and radiation, but with no vacuum energy, the curve of $a(t)$ has negative second derivative and thus $t_0 < 1/H_0$. Show that this is not always the case if there is a non-zero vacuum energy.

4. Consider a closed ($k = 1$) FRW theory with matter and vacuum energy, corresponding to positive cosmological constant Λ, and no radiation.

   (a) Show that, for any positive Λ, there is a critical value of the matter density ρ_m such that $a(t)$ is a constant. Find this critical value. Note that this metric is the Einstein static universe. This was, indeed, Einstein's original model.

   (b) What is the spatial volume of this universe in terms of Λ.

   (c) If ρ_m differs slightly from this critical value, so $a(t)$ varies with time, does the evolution remain close to the static universe, or diverge from it?