1/28/16 Lecture 7 outline

• Last time: one-loop self-energy for $\lambda\phi^4$:

$$-i\Pi'(p^2) = (-i\lambda)\tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \text{more loops.}$$

Going to Euclidean space, $d^4k = id^4k_E$,

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops.}$$

Recall expression $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. For $D=4$, get $\Omega_3 = 2\pi^2$, so

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{udu}{u+1} = \frac{\lambda m^2}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log(1+\frac{\Lambda^2}{m^2})\right).$$

Here Λ is a UV momentum cutoff. Result is quadratically (and also log) divergent as $\Lambda\to\infty$. The subject of renormalization is the physical interpretation of these divergences. The first thing to do is to regulate them, as we did above with a momentum cutoff. There are other ways to regulate. How one regulates is physically irrelevant. The physics is in the renormalization interpretation of the regulated results, and at the end of the day the choice of regulator doesn't matter.

• Casimir force example (see Schwartz ch 15). The $\frac{1}{2}\hbar\omega_k$ normal ordering constant (CC contribution) that we dropped can lead to an observable effect for quantum fields in a box. Consider two plates separated by a distance a. Put the whole system in a box of length L. E.g. in 1d, $\omega_n = n\pi/r$

$$E_{tot}(a) = E(a) + E(L-a) = (\frac{1}{a} + \frac{1}{L-a})\frac{\pi}{2}\sum_{n=1}^{\infty} n.$$

Several ways to regulate the divergent sum, e.g. the Riemann ζ function, $\zeta(s) = \sum_n n^{-s}$, analytically continued to $s\to -1$, $\zeta(-1) = -1/12$, i.e. $\sum_{n=1}^{\infty} n = -1/12$. All regulators give the same answer! $F(a) = -dE_{tot}/da = -\pi\hbar c/24a^2$. In 3d, with two photon polarizations, get $F(a) = -\pi^2\hbar cA/240a^4$, where A is the area of the walls (it's a pressure) and a is their separation. Predicted in 1948, observed in 1997.

• Study more generally the *superficial* degree of divergence of 1PI diagrams. Consider the general form of $\Gamma^{(n)}$:

$$\Gamma^{(n)} \sim \int \prod_{i=1}^{L}\frac{d^4k_i}{(2\pi)^4}\prod_{j=1}^{I}\frac{1}{l_j^2 - m^2 + i\epsilon}$$

For large k the integrand behaves as $\sim k^{4L-2I}$. Degree of UV divergence (superficially) is $D = 4L - 2I = 2I - 4V + 4$ (recall that $L = I - V + 1$). Suppose interaction is ϕ^p, then $pV = 2I + n$.

E.g. for $\lambda\phi^4$, $p = 4$, get $D = 4 - n$. This fits with what we found for $n = 2$, there was a quadratic divergence, i.e. $D = 2$. For $n = 4$, we get $D = 0$, which means a log divergence. For $n > 4$, we get $D < 0$, which means that there is no divergence at all (superficially, at least)! So the only two divergent cases are $n = 2$ and $n = 4$. The point will be that we can absorb these two divergent cases into corrections to the two parameters m and λ. That is the statement that the theory for $p = 4$ is renormalizable.

For $p = 6$, write $4V_4 + 6V_6 = 2I + n$, get $D = 4 - n + 2V_6$. The V_4 vertex is renormalizable, the V_6 is not. For $\lambda\phi^4$, the UV divergent terms are $n = 2, 4$. Higher n diagrams only have sub-divergences, which will be accounted for by properly treating the $n = 2$ and $n = 4$ cases. Example of a $n = 6$ diagram with a sub-divergence from the $n = 2$ diagram. Contrast $\lambda_4\phi^4$ with a $\lambda_3\phi^3$ theory (super-renormalizable) and a $\lambda_6\phi^6$ theory (non-renormalizable).

More generally, with bosons and fermions, $D = \sum_i n_i d_i + 2(IB) + 3(IF) - 4\sum_i n_i + 4$, where n_i is the number of vertices of i-th type and d_i is the number of derivatives in that interaction, and IB and IF are the numbers of internal boson and fermion lines. Then $D = -B - \frac{3}{2}F + 4 + \sum_i(\text{dim}\mathcal{L}_i - 4)$, where B and F are the numbers of external bose and fermion lines.

• Dimensional analysis and understanding the degrees of divergence by power-counting. In $\hbar = c = 1$ units, dimensionful quantities can be written as $x \sim m^{[x]}$, which defines $[x]$, the mass dimension of x. In particular, in D space-time dimensions, we have $[S] = 0$ and $[d^D x] = -D$, so $[\mathcal{L}] = D$ so scalars have $[\phi] = (D-2)/2$ and fermions have $[\psi] = (D-1)/2$. We see that a $\lambda_p\phi^p$ theory has $[\lambda_p] = D - p(D-2)/2$. In particular, for $D = 4$, get $[\lambda_p] = 4 - p$, showing why $p = 4$ is special, as compared with say $\lambda_3 \sim M$ and $\lambda_6 \sim M^{-2}$. Since $\Gamma^{(n)}$ has units of action, i.e. $\hbar$, it has $[\Gamma^{(n)}] = 0$. So a contribution with e.g. V_6 vertices has, on dimensional grounds, a factor of $(\lambda_6 E^2)^{V_6}$, where E is some energy scale. This reproduces the degree of UV divergence if we take $E \sim \Lambda \to \infty$. Discuss similar power counting for gravity, and for Fermi's 4-fermion weak-interaction vertex. Interpretation as low-energy effective theory with cutoff. "Non-renormalizable" theories are fine, and actually nice, in the IR, and just need some fixing up in the UV, but some UV completion.

General integrals

$$I_n(a) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a)^n}$$

with n integer and $Im(a) > 0$ and k in Minkowski space. See

$$I_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} I_1(a), \qquad I_1 = \frac{-i}{16\pi^2} \int_0^{\Lambda^2} du \frac{u-a+a}{u-a}$$

where we used the solid angle $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$, which is $2\pi^2$ for $D = 4$. Get

$$I_n(a) = i\left(16\pi^2(n-1)(n-2)a^{n-2}\right)^{-1} \qquad \text{for} \quad n \geq 3.$$

Special cases

$$I_1 = \frac{i}{16\pi^2} a \ln(-a) + \ldots,$$

$$I_2 = \frac{-i}{16\pi^2} \ln(-a) + \ldots,$$

where ... are terms involving the regulator.