1/19/16 Lecture 5 outline

- Everything in this bullet is review from the last lecture:

$$Z[J(x)] = \int [d\phi] \exp(\frac{i}{\hbar}\int d^4x[\mathcal{L} + J(x)\phi(x)]) = N \exp[\frac{i}{\hbar}S_{int}[-i\hbar\frac{\delta}{\delta J}])Z_{free}[J], \quad (1)$$

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x)D_F(x-y)J(y)). \quad (2)$$

$$iW[J] \equiv \ln Z[J]$$

$iW[J]$ is the generating functional for connected Green's functions

$$G^{(n)}_{conn}(x_1, \ldots x_n) = \prod_{j=1}^{n} \frac{-i\hbar\delta}{\delta J(x_j)} iW[J],$$

i.e.

$$iW[J] = \sum_{n=1}^{\infty} \frac{(i/\hbar)^n}{n!} \int d^4x_1 \ldots d^4x_n G^{(n)}_{conn}(x_1, \ldots x_n) J(x_1) \ldots J(x_n).$$

The 1-point function (with source J left non-zero):

$$G^{(1)}_{conn}(x) = (-i\hbar)\frac{\delta iW}{\delta J} = (-i\hbar)\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \equiv \overline{\phi}(x),$$

Picture this as a propagator connecting the point x to a blob, where the blob represents all diagrams from expanding in the interaction, and including connecting to the external source $J(x)$ (before setting it to zero). The denominator cancels off the disconnected vacuum bubble diagrams.

$$G^{(2)}_{conn}(x,y) = (-i\hbar)^2 \frac{\delta^2}{\delta J(x)\delta J(y)}(iW) = \langle \phi(x)\phi(y)\rangle_J - \langle \phi(x)\rangle_J \langle \phi(y)\rangle_J.$$

The first term includes both connected and disconnected contributions, and the 2nd term precisely cancels off the disconnected ones. Similarly $\delta W/\delta J^3$ has terms like $\langle \phi\phi\phi\rangle - (\langle\phi\phi\rangle\langle\phi\rangle + 2\ terms) + 2\langle\phi\rangle\langle\phi\rangle\langle\phi\rangle$, which give precisely $\langle\phi\phi\phi\rangle_{connected}$. Can prove by induction that the *log* in W properly subtracts away all non-connected diagrams.

- Example: free Klein Gordon theory. We found

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x)D_F(x-y)J(y)), \quad (3)$$

so we get

$$iW_{free}[J] = -\tfrac{1}{2}\hbar^{-1}\int d^4x \int d^4y J(x)D_F(x-y)J(y).$$

The only connected Green's function in this free case are the 1-point and 2-point functions:

$$G^{(1)}_{conn,free}(x) = \bar{\phi}_{free}(x) = -i\hbar \frac{\delta}{\delta J(x)} iW_{free} = i\int d^4y D_F(x-y)J(y),$$

$$G^{(2)}_{conn,free}(x,y) = (-i\hbar)^2 \frac{\delta^2}{\delta J(x)\delta J(y)} iW_{free} = \hbar D_F(x-y).$$

Note that the propagator contains a factor of $\hbar$. In an interacting theory, like $\lambda\phi^4$,

$$G^{(2)}(x,y) = \hbar D_F(x-y) + O(\lambda) \text{ corrections}.$$

In an interacting theory, the vertices have factors like $-i\lambda/\hbar$, while the propagators are proportional to $\hbar$. Suppose a diagram has I internal lines, V vertices, L loops. Connected graphs have $L = I - V + 1$. Graphs go like $\hbar^{-V}\hbar^{I} = \hbar^{L-1}$. So

$$W[J] = W_{-1}\hbar^{-1} + W_0 + \hbar W_1 + \ldots,$$

where W_{-1} are tree-graphs, W_0 gives the 1-loop graphs, $W_{\ell-1}$ gives ℓ-loop graphs.

• Emphasize that tree graphs are classical. Example: consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \phi J$, with the source term J. The classical field EOM is

$$(\partial_\mu\partial^\mu + m^2)\phi_c = -\frac{1}{3!}\lambda\phi_c^3 + J(x).$$

We can solve this classically to zero-th order in λ using $(\partial_\mu\partial^\mu + m^2)D_F(x-y) = -i\delta(x-y)$:

$$\phi_c^{(0)}(x) = \int d^4y D_F(x-y) iJ(y).$$

To solve to next order in λ, we plug this back into the above:

$$\phi_c^{(1)}(x) = \phi_c^{(0)}(x) - i\frac{1}{3!}\lambda \int d^4y D_F(x-y)\phi_c^{(0)}(y)^3$$

Continue this way, to obtain $\phi_c(x) = \sum_{n=0}^{\infty}\phi_c^{(n)}$, where $\phi_c^{(n)}(x) \sim \lambda^n$, this can be represented as a sum of tree-level diagrams, with one ϕ and different numbers of J's on the external legs. This is perturbation theory for the classical field theory. We solve $\frac{\delta}{\delta\phi}(S[\phi] + \int J\phi)|_{\phi=\phi_c} = 0$ for $\phi_c[J]$, and plug it back in to the action and source term, to get $W_{-1}[J] = S[\phi_c] + \int \phi_c J$. The LHS depends on J but not ϕ_c, since we solve for ϕ_c by $\frac{\delta}{\delta\phi_c}W_{-1}[J] = 0$. Likewise, $S[\phi_c]$ does not depend on J. This is a Legendre transform:

$$W_{-1}[J] = S[\phi_c] + \int \phi_c J, \qquad \phi_c = \frac{\delta}{\delta J}W_{-1}[J], \qquad J = -\frac{\delta}{\delta\phi_c}S[\phi_c]$$

which fits with $\frac{\delta}{\delta J}S[\phi_c]=0$. $\phi_c=\frac{\delta}{\delta J}W_{-1}[J]$ is the $\hbar\to 0$ limit of $\overline{\phi}(x)\equiv\langle 0|\phi|0\rangle_J/\langle 0|0\rangle_J$.

$\phi_c(x)$ and $J(x)$ are Legendre transform conjugate variables. Recall e.g. in thermodynamics, $dE=TdS-PdV$, so $E=E(S,V)$, and the conjugate variables are $T\leftrightarrow S$ and $P\leftrightarrow V$, we can define e.g. $E+PV=H(S,P)$, so adding PV to E changes it from being a function of V to being a function of P, with $P=-\partial E/\partial V$ and $V=\partial H/\partial P$. Likewise, above, for $S[\phi_c]$ vs $W_{-1}[J]$.

• We now extend this to $\hbar\neq 0$, i.e. the full diagrams including loops:

$$W[J]=\Gamma[\overline{\phi}]+\int d^4xJ(x)\overline{\phi}(x).$$

A Legendre transform, with the inverse transform:

$$\Gamma[\overline{\phi}]=W[J]-\int d^4xJ(x)\overline{\phi}(x).$$

(The quantities here are not operators.) And

$$\overline{\phi}(x)=\frac{\delta W[J]}{\delta J(x)}=\frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J},\qquad J=-\frac{\delta}{\delta\phi}\Gamma[\phi].$$

Here $\Gamma[\phi]$ is the *quantum effective action*, with

$$\lim_{\hbar\to 0}\Gamma[\phi]=S_{classical}[\phi_c].$$

As we will discuss, the point is that $W[J]$, which contains all connected diagrams, including loops, can be obtained by *tree-level* diagrams, provided we replace the classical action with the quantum effective action. This will be very useful when we start renormalization.

• Aside. We have seen that the loop expansion is an expansion in powers of $\hbar$, since diagrams go like $\hbar^{L-1}$. Question: are we expanding in $\hbar$ (loops), or in powers of the small coupling constants, or both? Answer: it's generally the same expansion. Consider e.g. $\lambda\phi^r$ interaction. Then a connected diagram with E external lines (amputating their propagators) and I internal lines and V vertices is $\sim\hbar^{I-V}\lambda^V$. Now we use $L=I-V+1$ and $E+2I=rV$ (conservation of ends of the lines) to get that the diagram is $\sim\left(\hbar\lambda^{2/(r-2)}\right)^{L-1}\lambda^{E/(r-2)}$, so for fixed E the loop expansion is an expansion in powers of the effective coupling $\alpha\sim\hbar\lambda^{2/r-2}$.

• We can also think of the effective action, $\Gamma[\phi]$, as a generating functional:

$$\Gamma[\phi]=\sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1\dots d^4x_n\Gamma^{(n)}(x_1,\dots x_n)\phi(x_1)\dots\phi(x_n).$$

Again,

$$\Gamma[\phi] = \frac{1}{\hbar}\left(S[\phi] + \mathcal{O}(\hbar)\right).$$

The terms that $\Gamma[\phi]$ generates are called the 1-particle irreducible diagrams,

$$\Gamma^{(n)}(x_1, \dots x_n) = \langle T\phi(x_1)\dots\phi(x_n)\rangle|_{1PI}.$$

The definition is that the diagrams are connected, and moreover remains connected upon removing any one internal progagator (and amputating all external legs).

• We usually compute them in momentum space, taking all external momenta to be incoming, related to above $\Gamma^{(n)}$ by Fourier transform.

$$\text{1PI diagram} \equiv i\tilde{\Gamma}^{(n)}(p_1, \dots p_n),$$

where the external propagators are amputated, and the $(2\pi)^4\delta^4(\sum_i p_i)$ is omitted. If there is an interaction like $V = g\phi^n/n!$, then, at tree-level, $\tilde{\Gamma}^{(n)} = -g$.

• Draw some examples of momentum space $n = 2, 4, 6$ point 1PI diagrams in $\lambda\phi^4$, taking all external momenta to be incoming.

• It is convenient to use a special definition for case $n = 2$: define the 1PI diagram sum to be $-i\Pi'(p)$. Get the full 2-point function by summing the geometric series of 1PIs:

$$D(p) = \frac{i}{p^2 - m^2 + i\epsilon}\sum_{n=0}^{\infty}\left[(-i\Pi'(p^2))\frac{i}{p^2 - m^2 + i\epsilon}\right]^n = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon}.$$

For $n = 2$ we define

$$i\tilde{\Gamma}^{(2)}(p, -p) \equiv \text{1PI diagram} + i(p^2 - m^2) = i(p^2 - m^2 - \Pi'(p^2))$$

So then

$$D(p) = \frac{i}{\tilde{\Gamma}^{(2)}} = \frac{i}{p^2 - m^2 - \Pi'(p^2)}.$$

$\Pi'(p^2)$ is called the self-energy, like momentum dependent mass term. The special definition of $\tilde{\Gamma}^{(2)}$ is because $D(p) = i/\tilde{\Gamma}^{(2)}$ will be nice, and allow extending to higher point functions.

• The point of the 1PI diagrams is that the quantum loop corrections are simply obtained by replacing the vertices with the 1PI greens functions! Indeed, Draw pictures for $n = 2, 4, 6$ point functions. Obtain the full $W[J]$ via tree-graphs assembled from the 1PI building blocks.

• Note that there are are no tree level IPI diagrams for $\tilde{\Gamma}^{(n)}$ except for $n = 4$ in $\lambda\phi^4$, so $\tilde{\Gamma}^{(n)} = -\lambda\hbar^{-1}\delta_{n,4} + \mathcal{O}(\hbar^0) + \dots$. At order $\hbar^0$, i.e. 1-loop, note that there are terms for all even n. (There can not be terms for odd n, because of the $\phi \to -\phi$ symmetry.)