1/14/16 Lecture 4 outline

• Last time,

$$Z[J(x)] = \int [d\phi] \exp(\frac{i}{\hbar}\int d^4x[\mathcal{L} + J(x)\phi(x)]) = N\exp[\frac{i}{\hbar}S_{int}[-i\hbar\frac{\delta}{\delta J}])Z_{free}[J], \qquad (1)$$

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x) D_F(x-y) J(y)). \qquad (2)$$

The green's functions are then given by

$$\begin{aligned} G^{(n)}(x_1 \dots x_n) &= \frac{\int [d\phi]\phi(x_1)\dots\phi(x_n)\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}{\int [d\phi]\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}, \\ &= \frac{1}{Z[J]}\prod_{j=1}^{n}\left(-i\hbar\frac{\delta}{\delta J(x_j)}\right)\cdot Z[J]\big|_{J=0}. \end{aligned}$$

E.g. for $\lambda\phi^4/4!$ get

$$G^{(n)}(x_1, \dots x_n) = \frac{1}{Z[J]}\prod_{j=1}^{n}(-i\hbar\frac{\delta}{\delta J(x_j)})\sum_{N=1}^{\infty}\frac{1}{N!}\left(-i\frac{\lambda}{4!\hbar}\int d^4y(-i)^4\frac{\delta^4}{\delta J(y)^4}\right)^N Z_0[J]\big|_{J=0}.$$

Consider, for example, the 4-point function $G^{(4)}(x_1, x_2, x_3, x_4) \equiv \langle T\phi(x_1)\dots\phi(x_4)\rangle/\langle 0|0\rangle$. Draw diagrams. We can choose to normalize $Z[J=0]=1$, since we anyway divide by the vacuum-to-vacuum amplitude, to cancel the bubble diagrams. For computing S-matrix elements, we will especially be interested in *connected* Green's functions. There are nice combinatoric formulae. E.g.

$$\sum \text{all diagrams} = \left(\sum \text{“connected”}\right)\cdot\exp(\sum \text{disconnected vacuum bubbles}).$$

And the vacuum bubble diagrams cancel. We write “connected” because for $n > 2$ point functions there are still disconnected diagrams, connected to the external points, included in this sum. We will not need these either, and will shortly discuss how to eliminate them.

• We got the functional integral to converge via the $i\epsilon$, recall it came from evaluating the gaussian functional integrals like $\int_{-\infty}^{\infty} d\phi e^{ia\phi^2}$, taking a =real $+i\epsilon$. There is another way, which is often very useful: Wick rotate to Euclidean space. The k_0 momentum integral, like that in

$$D_F(x) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}$$

can be analytically continued, as long as no poles are crossed. We can "Wick rotate" the dk_0 by $+\pi/2$, so k_0 runs from $-i\infty$ to $+i\infty$ along the imaginary axis. This allows continuation to $k_0 = ik_4$, with k_4 real, and k_4 runs from $-\infty$ to $+\infty$. So $k^2 = -k_E^2$, and $d^4k = id^4k_E$. To avoid having e^{ikx} blow up anywhere, we also continue time: $x_0 = -ix_4$, so $d^4x = -id^4x_E$. **Comment on signs:** we rotate $k_0 = e^{i\alpha}k_4$ with α running from $0 \to \pi/2$, and then to keep $e^{ik_0x_0}$ oscillatory we need to rotate by an opposite phase, $x_0 = e^{-i\alpha}x_4$. So $x_0 = -ix_4$ at the end. (Note however that this sign choice gives $k_0x_0 - k\vec{x} = k_4x_4 - \vec{k}\cdot\vec{x}$ is not 4d rotationally invariant.)

The Feynman propagator, in Euclidean space, is

$$\Delta_E(x) = \int \frac{d^4k_E}{(2\pi)^4} e^{-ikx} \frac{1}{k_E^2 + m^2},$$

where we can now drop the $i\epsilon$, since it's no longer needed. Note $k_E^2 + m^2$ is never zero, so the integrand never has a pole, and the solution Δ_E is unique.

The action changes as $S = \int d^4x(\frac{1}{2}\partial\phi\partial\phi - V) = i\int d^4x_E(\frac{1}{2}\partial_{x_E}\phi\partial_{x_E} + V) = iS_E$, where S_E looks like the energy now, $S_E =$ "H"! Then

$$\int [d\phi] \exp[\frac{i}{\hbar}S] \to \int [d\phi] e^{-\frac{1}{\hbar}\text{“}H\text{”}}$$

which is like the partition function of stat mech (as you saw in your HW)! (But here "H" is like the Hamiltonian of a theory living in 4 spatial dimensions..). Note $\hbar$ here appears as does T (temperaure) there, connects intuition of quantum fluctuations with intuition of thermal fluctuations!

It is sometimes useful to do all Feynman diagram computations in Euclidean space, and analytically continue back to Minkowski at the end of the day. So

	Mink	Euc
propagator	$\frac{i}{k^2-m^2} = \frac{-i}{k_E^2+m^2}$	$\frac{1}{k_E^2+m^2}$
vertex	$-ig$	$-g$
loop	$\int \frac{d^4k}{(2\pi)^4} = i\int \frac{d^4k_E}{(2\pi)^4}$	$\int \frac{d^4k_E}{(2\pi)^4}$.

Comparing with what we had before, we have dropped some factors of i:

$$i^{L+V-I} = i,$$

since (connected) diagrams have $L = I - V + 1$. So every diagram in the sum just differs by a factor of i, so the sums work the same as before (no relative differences).

• All disconnected diagrams drop out when we consider $S - 1$: they correspond to e.g. the 1. The LSZ reduction formula says that we get $S - 1$ from amputating the external propagators and going on shell, $\sim \prod_{i=1}^{n}(p_i^2 - m_i^2)\widetilde{G}(p_1, \ldots p_n)$, where the $p_i^2 - m_i^2 \to 0$ factors eliminate the external propagators. These factors also eliminate all contributions from disconnected Green's functions (draw pictures), since they have less than n external propagators. So, in the end, we're only interested in the fully connected diagrams. There is a generating functional for them for them. (N.B. sometimes people reverse the names of what I'm calling W and Z. Peskin calls $W \to E$.) Defining,

$$iW[J] \equiv \ln Z[J]$$

$iW[J]$ is the generating functional for connected Green's functions

$$G_{conn}^{(n)}(x_1, \ldots x_n) = \prod_{j=1}^{n} \frac{-i\hbar\delta}{\delta J(x_j)} iW[J],$$

i.e.

$$iW[J] = \sum_{n=1}^{\infty} \frac{(i/\hbar)^n}{n!} \int d^4x_1 \ldots d^4x_n G_{conn}^{(n)}(x_1, \ldots x_n) J(x_1) \ldots J(x_n).$$

In momentum space, we can write:

$$iW[J] = \sum_{n=1}^{\infty} \frac{(i/\hbar)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1) \ldots \tilde{J}(-k_n) \tilde{G}_{conn}(k_1, \ldots k_n).$$

- Examples, to illustrate how $iW[J] \equiv \ln Z[J]$ gives the connected diagrams. First consider the 1-point function

$$G_{conn}^{(1)}(x) = (-i\hbar)\frac{\delta iW}{\delta J} = (-i\hbar)\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J} \equiv \overline{\phi}(x),$$

where $\overline{\phi}(x)$ is not an operator – it is the average of the quantum field. Sometimes it is called ϕ_{cl}, since it behaves like a classical field. But it has quantum effects built in. Picture this diagrammatically as a propagator connecting the point x to a blob, where the blob represents a $\sum_n \lambda^n$ sum of diagrams. There are no disconnected diagrams, thanks to the denominator above which subtracts out the disconnected vacuum bubble diagrams.

Now consider the two point function

$$G^{(2)}(x, y) = (-i\hbar)^2 \frac{\delta^2}{\delta J(x)\delta J(y)}(iW) = \langle \phi(x)\phi(y)\rangle_J - \langle \phi(x)\rangle_J \langle \phi(y)\rangle_J.$$

Note that $\langle \phi(x)\phi(y)\rangle$ has two types of contributions, connected and disconnected; the 2nd term precisely cancels off the disconnected ones. The connected one is pictured as a line connecting x and y, with a single blob propagator, whereas the disconnected contribution has two disconnected blobs. Similarly $\delta W/\delta J^3$ has terms like $\langle\phi\phi\phi\rangle - (\langle\phi\phi\rangle\langle\phi\rangle + 2\ terms) + 2\langle\phi\rangle\langle\phi\rangle\langle\phi\rangle$, which give precisely $\langle\phi\phi\phi\rangle_{connected}$. Can prove by induction that the *log* in W properly subtracts away all non-connected diagrams!