1/12/16 Lecture 3 outline

• Last time,

$$Z[J(x)] = \int [d\phi] \exp(\frac{i}{\hbar}\int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$G^{(n)}(x_1, \ldots x_n) = \langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle/\langle 0|0\rangle = Z[J]^{-1}\prod_{j=1}^{n}\left(-i\hbar\frac{\delta}{\delta J(x_i)}\right)Z[J]\big|_{J=0}.$$

E.g. for the QFT KG example, free scalar field theory $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - J\phi$ has generating functional

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4xd^4yJ(x)D_F(x-y)J(y)), \qquad (1)$$

with

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}.$$

• Let's double check the factors of $\hbar$ etc, considering an ordinary gaussian integral, $\int_{-\infty}^{\infty} d\phi e^{-b\phi^2 + c\phi} = \sqrt{\frac{\pi}{b}}e^{-c^2/4b}$, as seen by completing the square. We now replace $b \to \frac{1}{2i\hbar}(-\partial^2 - m^2 + i\epsilon)$ and $c \to \frac{i}{\hbar}J$, so $e^{-c^2/4b} \to e^{\frac{1}{2}(J/\hbar)^2(i\hbar)(-\partial^2 - m^2 - i\epsilon)^{-1}}$, or more precisely, the expression (1). As discussed last time, we compute green's functions from $Z[J]$ by acting with $\phi(x) \to \frac{\hbar}{i}\frac{\delta}{\delta J(x)}$. In particular, we get $G_0^{(2)}(x,y) = \hbar D_F(x-y)$. As another example, $G_0^{(4)}(x_1, x_2, x_3, x_4) = G_0^{(2)}(x_1, x_2)G_0^{(2)}(x_3, x_4)$+(2 permutations).

Of course, we can set $\hbar = 1$, and restore it at the end using dimensional analysis. But it's useful to keep $\hbar$, or the coupling constants, as a loop counting parameter. We'll discuss this soon. So for now let's keep $\hbar$ explicit.

• Sometimes we like to redefine our source, replacing $J \to \hbar J$. So we'll then write $\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \hbar J\phi$, and replace (1) with

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar\int d^4xd^4yJ(x)D_F(x-y)J(y)), \qquad (2)$$

with $\phi(x) \to \frac{1}{i}\frac{\delta}{\delta J(x)}$. Either way gives the same answers for the green's functions, of course, – it's just semantics for what we want to call the source.

• Now let's consider an interacting theory. Notice that

$$\int [d\phi] \exp(\frac{i}{\hbar}[S_{free} + S_{int}[\phi] + \hbar\int d^4xJ\phi]) = \exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J].$$

So

$$Z[J] = N \exp[\frac{i}{\hbar} S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J], \tag{3}$$

where N is an irrelevant normalization factor (independent of J). The green's functions are then given by

$$\begin{aligned} G^{(n)}(x_1 \dots x_n) &= \frac{\int [d\phi] \phi(x_1) \dots \phi(x_n) \exp(\frac{i}{\hbar} S_I[\phi]) \exp[\frac{i}{\hbar} S_{free}]}{\int [d\phi] \exp(\frac{i}{\hbar} S_I[\phi]) \exp[\frac{i}{\hbar} S_{free}]}, \\ &= \frac{1}{Z[J]} \prod_{j=1}^{n} \left(-i\hbar \frac{\delta}{\delta J(x_j)} \right) \cdot Z[J] \big|_{J=0}. \end{aligned}$$

(The denominator (in both lines) cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function.)

- Illustrate the above formulae, and relation to Feynman diagrams, e.g. $G^{(1)}$, $G^{(2)}$ and $G^{(4)}$ in $\lambda\phi^4$ theory. The functional integral accounts for all the Feynman diagrammer; even symmetry factors etc. come out simply from the derivatives w.r.t. the sources, and the expanding the exponentials,

$$G^{(n)}(x_1, \dots x_n) = \frac{1}{Z[J]} \prod_{j=1}^{n} (-i \frac{\delta}{\delta J(x_j)}) \sum_{N=1}^{\infty} \frac{1}{N!} \left(-i \frac{\lambda}{4!\hbar} \int d^4 y (-i)^4 \frac{\delta^4}{\delta J(y)^4} \right)^N Z_0[J] \big|_{J=0}.$$

etc. Consider, for example, the 4-point function $G^{(4)}(x_1, x_2, x_3, x_4) \equiv \langle T\phi(x_1) \dots \phi(x_4) \rangle / \langle 0|0 \rangle$ in $\frac{\lambda_4}{4!}\phi^4$. So take 4-fuctional derivatives w.r.t. the source, at points $x_1 \dots x_4$, i.e. $\delta/\delta J(x_1) \dots \delta/\delta J(x_4)$. The $\mathcal{O}(\lambda^0)$ term thus comes from expanding the exponent in (2) to quadratic order (4 J's), corresponding to the disconnected diagrams with two propagators. Each propagator ends on a point x_i. This is like the 4-point function in the SHO homework. Now consider the $\mathcal{O}(\lambda)$ contribution, coming from expanding out the interaction part of the exponent in (3) to $\mathcal{O}(\lambda)$. There are now 4 extra $\delta/\delta J(y)$, for a total of 8, so the contributing term comes from expanding the exponent in (2) to 4-th order, i.e. there are 4 propagators. This gives the connected term, along with several disconnected terms. Go through these terms and their combinatorics.