1/7/16 Lecture outline

• Gaussian integrals:

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_i + \tilde{J}_i\phi_i) = \pi^{N/2}(\det B)^{-1/2}\exp(B^{-1}_{ij}\tilde{J}_i\tilde{J}_j/4).$$

Evaluate via completing the square: the exponent is $-(\phi, B\phi) + (\tilde{J}, \phi) = -(\phi', B\phi') + \frac{1}{4}(\tilde{J}, B^{-1}\tilde{J})$, where $\phi' = \phi - \frac{1}{2}B^{-1}\tilde{J}$. Again, we can similarly evaluate Gaussian integrals with phases in the exponent by analytic continuation

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(\frac{i}{\hbar}(\tfrac{1}{2}A_{ij}\phi_i\phi_i + J_i\phi_i)) = (2\pi i\hbar)^{N/2}(\det A)^{-1/2}\exp(-iA^{-1}_{ij}J_iJ_j/2\hbar).$$

replacing $B \to -i(\frac{1}{2}A + i\epsilon)/\hbar$ and redefining $\tilde{J} \to iJ/\hbar$ for later convenience.

• Last time: derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t'|q, t\rangle = \int\int \prod_{j=1}^{N} dq_j \langle q'|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-iH\delta t}|q_{N-2}\rangle \dots \langle q_1|e^{-iH\delta t}|q\rangle,$$

where we'll take $N \to \infty$ and $\delta t \to 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]+\dots}$, we're not going to have to worry about this for $\delta t \to 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m}e^{-i\delta t V(q)}e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2|e^{-iH\delta t}|q_1\rangle = \int dp_1 \langle q_2|e^{-i\delta t p^2/2m}|p_1\rangle\langle p_1|e^{-iV(q)\delta t}|q_1\rangle,$$

$$= \int dp_1 e^{-iH(p_1,q_1)\delta t}e^{ip_1(q_2-q_1)}.$$

This leads to

$$\langle q', t'|q, t\rangle = \int [dq(t)][dp(t)]\exp(i\int_t^{t'} dt(p(t)\dot{q}(t) - H(p, q))).$$

Taking $H = p^2/2m + V(x)$, we do the p gaussian integral, with $A = -1/m$ and $J = \dot{q}$, so $\exp(-iA^{-1}_{ij}J_iJ_j/2\hbar) \to \exp(i\int dt\frac{1}{2}m\dot{q}^2)$, and we recover the Feynman path integral.

• The same derivation as above leads to e.g.

$$\langle q_4, t_4|T\widehat{q}(t_3)\widehat{q}(t_2)|q_1, t_1\rangle = \int [dq(t)]q(t_3)q(t_2)e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). E.g.

$$\langle\phi_b(\vec{x},T)|e^{-iHT}|\phi_a(\vec{x},0)\rangle = \int [d\phi]e^{iS/\hbar} \qquad S = \int d^4x\mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle\Omega|T\prod_{i=1}^n \phi_H(x_i)|\Omega\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^n \phi(x_i)\exp(iS/\hbar),$$

with $Z_0 = \int [d\phi]\exp(iS/\hbar)$. Again, as noted above, the T ordering will be automatic.

• Now introduce **sources** for the fields as a trick to get the time order products from derivatives of a generating function (or functional).

• Consider QM with Hamiltonian $H(q,p)$, modified by introducing a source for q, $H \to H - J(t)q$. (We could also add a source for p, but don't bother doing so here.) Consider moreover replacing $H \to H(1-i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_J = \int [dq]\exp[i\int dt(L + J(t)q)/\hbar] \equiv Z[J(t)].$$

Once we compute $Z[J(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[J]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0|\prod_{j=1}^n Tq(t_j)|0\rangle = \prod_{j=1}^n \frac{1}{i}\frac{\delta}{\delta J(t_j)}Z[f]\big|_{f=0},$$

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta J(t)}J(t') = \delta(t-t')$.

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture. We'll be interested in such generating functionals, and their generalization to quantum field theory (replacing $t \to (t, \vec{x})$).

• We'll want to compute amplitudes like

$$\frac{\langle 0|\prod_i Tq(t_i)|0\rangle_{J=0}}{\langle 0|0\rangle_{J=0}}$$

and for these the $\det A$ factor in the Gaussian integrals will cancel between the numerator and the denominator. This is related to the cancellation of vacuum bubble diagrams.

• Let's apply the above to compute the generating functional for the example of QM harmonic oscillator (scaling $m = 1$),

$$Z[J(t)] = \int [dq(t)] \exp(-\frac{i}{\hbar}\int dt \left[\tfrac{1}{2}q(t)(\frac{d^2}{dt^2} + \omega^2)q(t) - J(t)q(t)\right]).$$

This is analogous to the multi-dimenensional gaussian above, where i is replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix A_{ij} is replaced with the differential operator $A \to -(\frac{d^2}{dt^2} + \omega^2 - i\epsilon)$, where the $i\epsilon$ is to damp the gaussian, as mentioned above. Doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want, so

$$\frac{\langle 0|0\rangle_J}{\langle 0|0\rangle_{J=0}} = \text{“}\exp[-i\tfrac{1}{2}A^{-1}_{ij}J_iJ_j/\hbar]\text{”} = \exp[-\tfrac{1}{2}\hbar\int dtdt' J(t)G(t-t')J(t')],$$

with $G(t)$ the Green's function for the oscillator, $(-\partial_t^2 - \omega^2 + i\epsilon)G(t) = i\delta(t)$,

$$G(t) = \int_{-\infty}^{\infty}\frac{dE}{2\pi\hbar}\frac{i \quad e^{-iEt/\hbar}}{E^2/\hbar^2 - \omega^2 + i\epsilon} = \frac{1}{2\omega}e^{-i\omega|t|}. \tag{1}$$

The $i\epsilon$ here does the same thing as in the Feynman propagator: the pole at $E = \hbar\omega$ is shifted below the axis and that at $E = -\hbar\omega$ is shifted above. Equivalently, we can replace $E \to E(1 + i\epsilon)$, to tilt the integration contour below the $-\omega$ pole and above the $+\omega$ pole. Note then that $e^{-iEt/\hbar} \to e^{-iEt/\hbar}e^{Et\epsilon/\hbar}$, which projects on to the vacuum for $t \to \infty$ (the $i\epsilon$ projects on to the vacuum in the far future and also the far past).

For $t > 0$, the E contour is closed in the LHP and the residue is at $E = \hbar\omega$, while for $t < 0$ the contour is closed in the UHP, with residue at $E = -\hbar\omega$.

• Now that we know the generating functional, we can use it to compute time ordered expectation values via

$$\langle 0|T\prod_{i=1}^{n}\phi_H(t_i)|0\rangle/\langle 0|0\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^{n}\phi(t_i)\exp(iS/\hbar) = Z_0^{-1}\prod_{i=1}^{n}\frac{\hbar}{i}\frac{\delta}{\delta J(t)}|_{J=0}.$$

with $Z_0 = \int [d\phi]\exp(iS/\hbar)$.

• On to QFT and the Klein-Gordon theory,

$$Z_0 = \int [d\phi]e^{iS/\hbar} \qquad S = \tfrac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term. This is completely analogous to our QM SHO example, simply replacing $\frac{d^2}{dt^2} + \omega^2 - i\epsilon$ there with $\partial^2 + m^2 - i\epsilon$ here – again, the $i\epsilon$ is to make the oscillating gaussian integral slightly damped. I.e. we should take $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $A \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar: it's the Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

As in the SHO QM example, we can compute field theory Green's functions via the generating functional

$$Z[J(x)] = \int [d\phi]\exp(i\int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle/\langle 0|0\rangle = Z[J]^{-1}\prod_{j=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right)Z[J]\big|_{J=0}.$$

E.g. for the KG example, $A = (-\partial^2 - m^2 + i\epsilon)$, so the generating functional is

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4xd^4yJ(x)D_F(x-y)J(y)), \qquad (2)$$

with $(-\partial^2 - m^2 + i\epsilon)D_F(x-y) = i\delta^4(x-y)$,

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}.$$

Can use this generating function to compute free field time ordered products, it reproduces Wick's theorem, Feynman diagrams. Again, S-matrix amplitudes are related (vis LSZ) to time-ordered products of fields. So, what we need to compute, are the Green's functions

$$G^{(n)}(x_i) \equiv \langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle/\langle 0|0\rangle.$$

Let's consider them in the free KG example. Find e.g. $G_0^{(2)}(x,y) = \hbar D_F(x-y)$, (where the subscript is to remind us it's the free theory), $G_0^{(4)} = G_0^{(2)}(x_1,x_2)G_0^{(2)}(x_3,x_4)+$ 2 permutations, etc.